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Research paper

Quasi-equal area subdivision algorithm for uniform points on a sphere with application to any geographical data distribution

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ABSTRACT

This paper describes a quasi-equal area subdivision algorithm based on equal area spherical subdivision to obtain approximated solutions to the problem of uniform distribution of points on a 2-dimensional sphere, better known as Smale's seventh problem. The algorithm provides quasi-equal area triangles, starting by splitting the Platonic solids into subsequent spherical triangles of identical areas. The main feature of the proposed algorithm is that the final adjacent triangles share common vertices that can be merged. It applies reshaping to the final triangles in order to remove obtuse triangles. The proposed algorithm is fast and efficient to generate a large number of points. Consequently, they are suitable for various applications requiring a large number of distributed points. The proposed algorithm is then applied to two geographical data distributions that are modeled by quasi-uniform distribution of weighted points.

1. Introduction

The problem of distributing N points uniformly over the surface of a sphere has been investigated for many decades (Robinson, 1961; Berman and Hanes, 1977; Mortari et al., 2011). This problem is one of the most challenging mathematical problems of the century and it is known as Smale's 7th problem (Smale, 1998). However, because of its implications in many areas of mathematics and its immediate practical applications in engineering, it has not only inspired mathematical researchers but also attracted the attention in various fields such as electrostatics, molecular structure, and crystallography (Saff and Kuijlaars, 1997). The capability of uniformly distributing points on a sphere has important theoretical consequences in old problems dating back to Thomson (1904) and Tammes problem (Tammes, 1930) and important applications such as survey sampling, optimization, dynamic modeling and information storage, and display in engineering, allowing the development of optimal algorithms (White, 2000; Mortari et al., 2011).

Various algorithms have been developed for a small number of points (Robinson, 1961; Berman and Hanes, 1977; Dragnev et al., 2002). However, most of them use optimization techniques that are not efficient for a large number of points. Other more modern algorithms, such as Chan's Quadrilateralized Spherical Cube Map (QSCM) projection (1975 Navy report, now out-of-print), extensively analyzed in the reference (O'Neill and Laubscher, 1976) and applied by Naval and NASA programs, and the algorithm by Snyder (1992), which is based on Platonic solids, are efficient and available. These methods all generate a total number of points (N) proportional to the number of faces of a Platonic solid; for instance, proportional to 6 (Cube or Hexahedron) for the QSCM. Teanby (2006) suggested an icosahedron-based method by subsequent quadrisection for evenly spaced binning data. Massey (2012) presented a method of constructing equal area triangles by repeatedly applying quadrisection to icosahedron and iterative equalization.

In Lee and Mortari (2013b) the authors introduced the main concepts developed in detail in this article. However, while Lee and Mortari (2013b) verified the proposed algorithms with Monte Carlo approach and the Smale's validation in the view of uniformity of distributing points, in the current manuscript the verification is not confined to the uniformity of distributing points, but to the subdivision method.

In view of this, the subdivision approach is considered to develop an algorithm to distribute a large number of points on the sphere. This paper is organized as follows. The first section of this paper provides the equations for the subdivision approach. Then, at the end of the original equal area subdivision algorithm the subsequent quasi-equal area final subdivision is provided. Finally, applications to geographical data are presented.

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2. Subdivision approach

2.1. Splitting a spherical triangle in two equal area spherical triangles

Consider the generic spherical triangle that is formed on the surface of the unit sphere by three great circular arcs intersecting pairwise in three vertices as shown in Fig. 1.

The area of a spherical triangle $[\hat{v}_A, \hat{v}_B, \hat{v}_C]$ is obtained by

$$S = A + B + C - \pi \tag{1}$$

where angles A, B, and C are the dihedral angles of the spherical triangle (Bronshtein et al., 2007).

Let a be the largest side angle, $\hat{v}_C \cdot \hat{v}_B = \cos a$. The problem to solve here is to find the point on the side a such that the two spherical triangles identified by the unit-vectors, $[\hat{v}_A, \hat{v}_B, \hat{v}_D]$ and $[\hat{v}_A, \hat{v}_C, \hat{v}_D]$, have identical areas. Since the splitting point, $\hat{v}_D$, is co-planar to $\hat{v}_C$ and $\hat{v}_B$, it can be linearly expressed by the unit-vectors $\hat{v}_C$ and $\hat{v}_B$ as follows.

$$\hat{v}_D = \frac{1}{\sin a}[\hat{v}_C \sin z + \hat{v}_B \sin(a - z)] \tag{2}$$

where z (see Fig. 1) is the side of the spherical triangle $[\hat{v}_A, \hat{v}_B, \hat{v}_D]$.

Now make use of x and y to denote the angles at the vertices of the spherical triangle $[\hat{v}_A, \hat{v}_B, \hat{v}_D]$. The area of the spherical triangle $[\hat{v}_A, \hat{v}_B, \hat{v}_D]$ is

$$S_1 = x + y + B - \pi = \frac{S}{2} = \frac{A + B + C - \pi}{2} \tag{3}$$

then

$$x + y = D = \frac{A + C + \pi - B}{2} \quad \text{and} \quad y = D - x \tag{4}$$

where D is not a new variable but a known quantity. Applying the law of cosines to the spherical triangle $[\hat{v}_A, \hat{v}_B, \hat{v}_D]$ gives

$$\cos y = \sin x \sin B \cos c - \cos x \cos B \tag{5}$$

Then, using the angle difference identity and Eq. (4), we obtain

$$\cos y = \cos D \cos x + \sin D \sin x = \sin x \sin B \cos c - \cos x \cos B \tag{6}$$

and

$$\tan x = \frac{\cos D + \cos B}{\sin B \cos c - \sin D} \quad \text{where } 0 < x < \frac{\pi}{2} \tag{7}$$

Finally, using the law of sines, $\sin z \sin y = \sin c \sin x$, with the spherical triangle $[\hat{v}_A, \hat{v}_B, \hat{v}_D]$

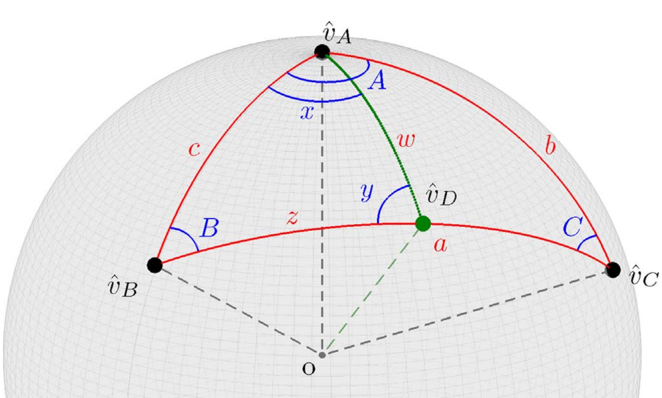


Fig. 1. Splitting a spherical triangle $[\hat{v}_A, \hat{v}_B, \hat{v}_C]$ in two equal area spherical triangles, $[\hat{v}_A, \hat{v}_B, \hat{v}_D]$ and $[\hat{v}_A, \hat{v}_C, \hat{v}_D]$. The angles at the vertices of the spherical triangle $[\hat{v}_A, \hat{v}_B, \hat{v}_C]$ are denoted by the upper case letters A, B, and C while the sides are denoted by lower-case letters a,b, and c. After subdivision x, y, and B are the dihedral angles and w, z, and c are the sides of the spherical triangles $[\hat{v}_A, \hat{v}_B, \hat{v}_D]$.

Table 1
Platonic solids parameters. v indicates the total number of vertices, e the total number of edges, f the total number of faces, p the number of edges in each face (3 for equilateral triangles, 4 for the squares, and 5 for regular pentagons), q the number of edges meeting at each vertex. The parameter s indicates the type of initial sub-division (3 for triSection 4 for quadrisection and 5 for pentasection) to create identical triangles and i=p f is the number of initial faces.

Platonic solids	v	e	f	p	q	s	i
Tetrahedron	4	6	4	3	3	3	12
Hexahedron	8	12	6	4	3	4	24
Octahedron	6	12	8	3	4	3	24
Dodecahedron	20	30	12	5	3	5	60
Icosahedron	12	30	20	3	5	3	60

$$\sin z = \frac{\sin x \sin c}{\sin(D - x)} \tag{8}$$

is obtained and the $\hat{v}_D$ can be computed using Eq. (2). The process can then be repeated by always splitting the longest side of the spherical triangles.

The idea of using spherical triangle splitting to generate points on a sphere finds the most natural starting point from the perfect spherical symmetry provided by Platonic solids. The parameters defining the five Platonic solids are summarized in Table 1 (Zwillinger, 2002). Since splitting a face into the number of edges with a center of face and vertices generates identical smaller triangles, initial division depends on shape of the face. Note that dual solids have same number of initial faces. Platonic solids with most initial faces are the dodecahedron and the icosahedron. For these solids the quasi-uniform distribution of points can be created by initially splitting the i=60 faces into 5 and 3 equal area triangles, respectively.

The sides of a Platonic solid can be projected onto a sphere where they form arcs. This “Platonic sphere” is the central projection of the sides of the Platonic solid onto the surface of a unit-radius sphere. The projection is on the Platonic solids' circum-sphere, which acts like a curved projection screen (Popko, 2012). All edges in Platonic solids have been transformed into geodesic arcs in corresponding platonic spheres. In platonic spheres all arcs have same length as well as all edges in Platonic solids. The vertices are corners in the case of spheres while the vertices are corner in the case of solids.

Let's show the procedure of equal-spherical area subdivision starting from an icosahedron. The vertices of an icosahedron can be defined using the Golden ratio

$$\varphi = \frac{1 + \sqrt{5}}{2} \tag{9}$$

The 12 vertices can then be obtained as all even permutations of the following set of coordinate triads

$$\left\{0,\ \pm\frac{1}{\sqrt{1 + \varphi^2}},\ \pm\frac{\varphi}{\sqrt{1 + \varphi^2}}\right\} \tag{10}$$

2.2. Equal-spherical area subdivision

For equal area subdivisions the algorithm must satisfy the following requirements:

(1) every subdivision generates triangles for recursive subdivision, and
(2) the greatest spherical dihedral angle cannot be greater than 90°. This does not allow triangles to degenerate.

It is possible to use various types of equal area subdivision which preserve area between faces in a planar triangle. However, a few

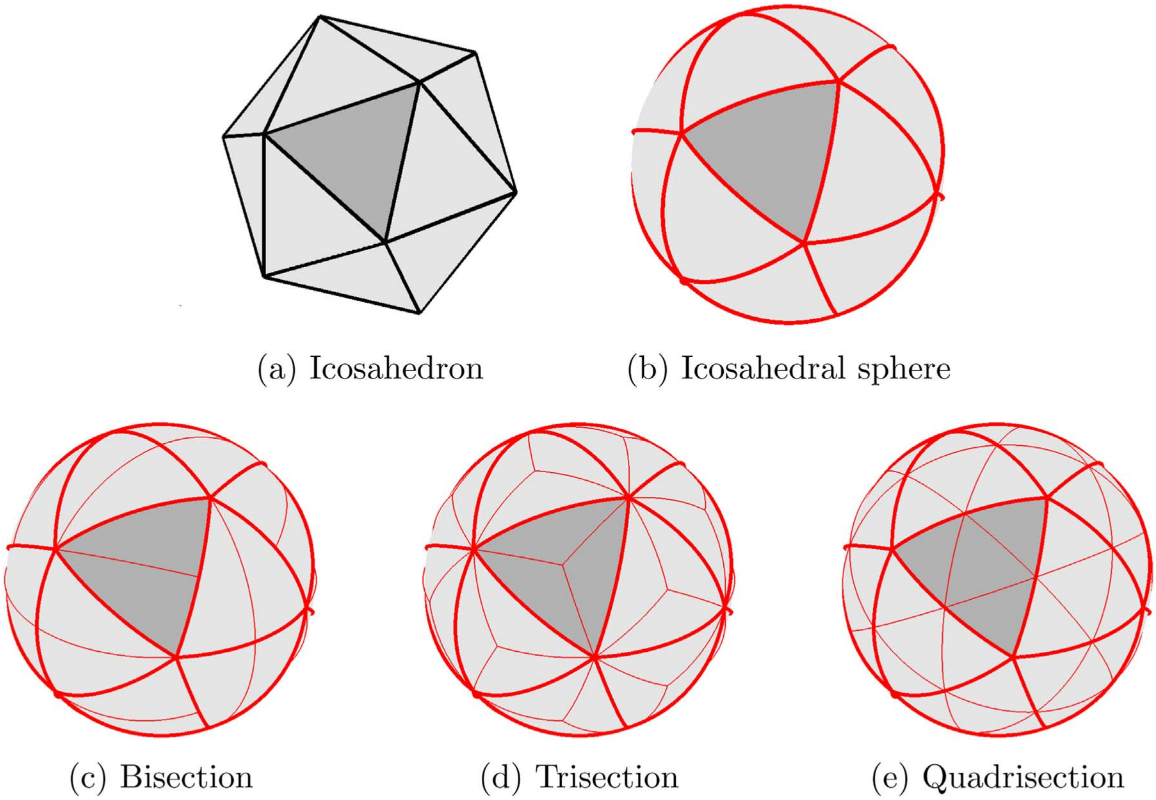


Fig. 2. Icosahedron and subdivision surfaces. (a) An icosahedron has 20 faces, 12 vertices, and 30 edges. (b) An icosahedral sphere also has same number of faces, vertices, and edges. However, faces are regular spherical triangles and edges are arc-edges. (c) Bisection can be used for recursive subdivision if the longest side is selected to split. (d) Trisection can be used as an initial subdivision. (e) Quadrisection is excluded in potential subdivision candidates since it cannot provide equivalent areas.

subdivision methods can be applied to a spherical triangle. Fig. 2c shows the bisection subdivision of spherical triangles. In particular, both planar and spherical faces are shown. In order to keep the sub-triangles as close as possible to the equilateral one, the longest side is selected to split. This avoids the generation of elongated triangles.

Three equal area subdivision (trisection) of spherical triangles can also be performed. For a generic spherical triangle, there is a unique point (direction) where the trisection can be done. However, the computation of this point/direction requires some effort and, more important, the subsequent trisection subdivisions do not avoid the creation of degenerated (elongated) triangles, which is in contradiction with requirement **(2)**. Fig. 2d shows the spherical faces of an initial trisection subdivision.

Four equal area subdivision (quadrisection) can be obtained by adding a new vertex at the midpoint of each edge of an equilateral triangle and dividing each edge in two. The quadrisection subdivision creates four new triangles in flat faces with equivalent areas (if the original triangle is equilateral). This four equal area subdivision is the method used by Teanby (2006) and Massey (2012). However, projecting these new triangles on the sphere does not provide equivalent spherical areas. Fig. 2e shows the result of projection on the first level of subdivision. In this figure the dark triangle is quadrisected in four triangles. The internal spherical triangle has a larger area than other adjacent three triangles. This difference then increases with subsequent quadrisections. In addition there is no freedom in choosing the midpoints along the edges for equivalent areas. Since it only works for planar triangles and not spherical triangles, Massey (2012) uses an iterative modification of the mesh to equalize the areas. Using this iterative method will equalize the areas at the expense of computing time and moving the points away from great circle diameters. For the above reasons quadrisection is excluded in potential subdivision candidates.

For all the above reasons, one method only can be used to recursively subdivide a spherical triangle satisfying requirements **(1)** and **(2)**: *subdivision of spherical triangles in two equal-area spherical triangles by splitting the longest side*.

3. Original equal-area subdivision algorithm

Starting with a Platonic solid (e.g., icosahedron) and subsequently performing a set of equal area triangle divisions (as previously described), a final set of small triangles, all with the same areas, are obtained. The original algorithm by Mortari et al. (2011) considers the directions to centers of these triangles as the set of quasi-uniform directions in space. The number of directions that can be obtained is dependent on the Platonic solid initially considered. For instance, starting from an icosahedron, a total number of $n = 20{\cdot}2^s$ directions can be obtained, where s is the number of subsequent divisions.

This original equal area subdivision algorithm creates quasi-uniform distributed points in space. The four initial subdivisions by this procedure are shown in Fig. 3. The number of times the equal area subdivision is performed (s) is subsequently referred to as the 'level'. Level 0 corresponds to the initial spherical icosahedron, level 1 refers to spherical triangles after the first subdivision, and so on. The final subdivision, unfortunately, creates adjacent spherical triangles that do not share vertices and obtuse adjacent triangles. To avoid this problem, after the final subdivision, every two adjacent elongated triangles are replaced by two more equilateral triangles. This single step reshaping procedure produces a final set of triangles with slightly different areas but greatly increases the uniformity of space distribution of the final set of triangles' centers. This improvement is described in the next section.

4. Quasi-equal area subdivision algorithm

The quasi-equal area subdivision algorithm scheme consists of two steps:

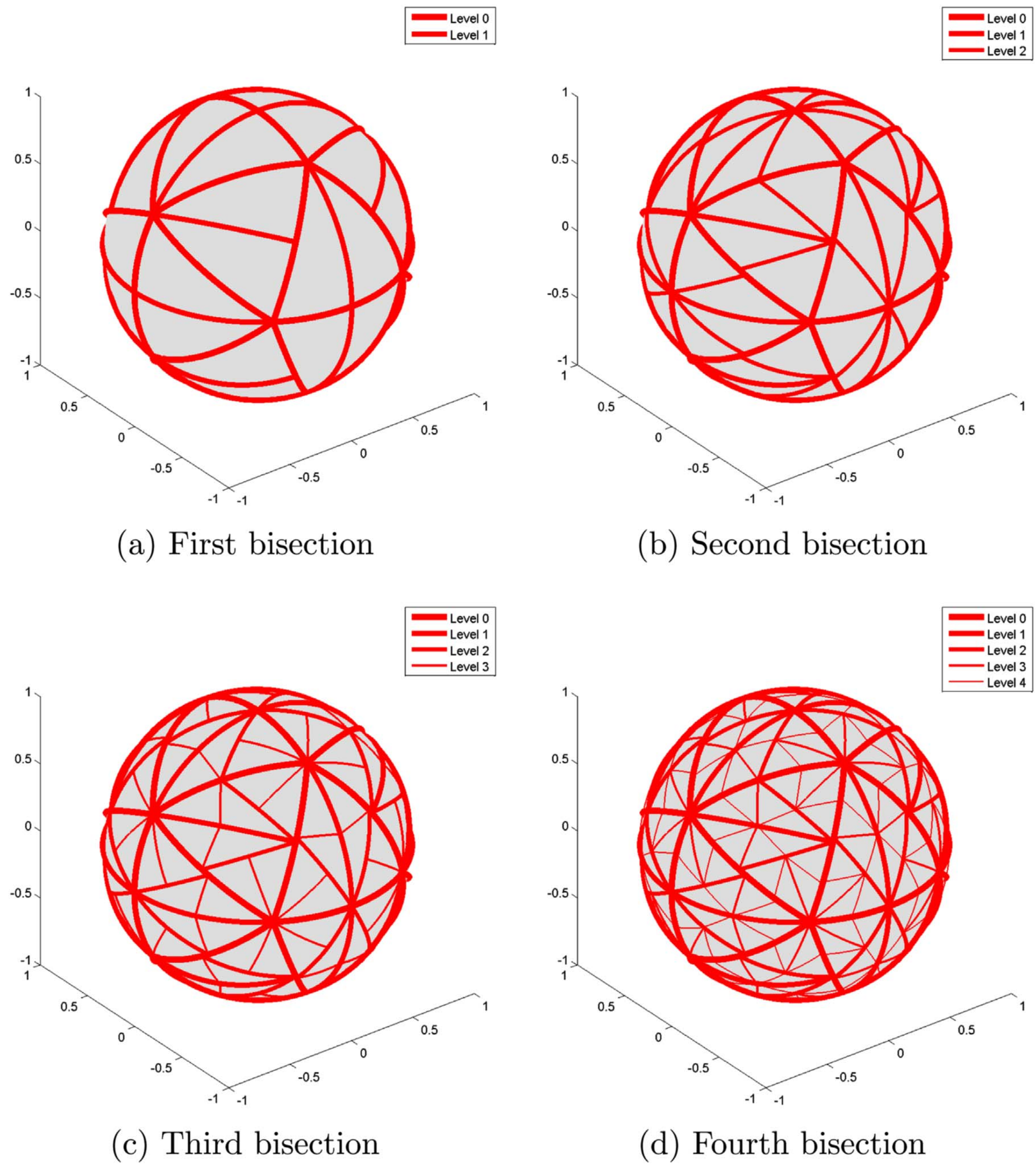


Fig. 3. The four initial equal-area subdivisions of an icosahedron using the original equal-area subdivision algorithm. The algorithm provides subsequent spherical triangles of identical areas using recursive bisections. The level is defined as the number of times bisection is performed.

1. dividing spherical triangles into equal area spherical triangles subsequently (perform a single equal area trisection, then perform *s* equal area subdivisions, as previously described), and
2. reshaping the final obtuse triangles after the final even number of divisions.

Note that reshaping (step 2) is applied one time only after all subsequent divisions, because *reshaping does not preserve the area of triangles*.

4.1. Step 1: Spherical triangles subdivision in equal-area spherical triangles

This subsequent subdivision consists of one trisection and several bisections. An icosahedron is first subdivided by the equal area trisection. There are now three triangles for each original triangle as shown in Fig. 4a. Note that this trisection makes adjacent triangles share common vertices after subdivision. Then, the trisection is followed by *s* sequential equal area bisections which are identical to subdivisions of the original equal area subdivision algorithm.

The number of vertices and faces of generated triangles in each level is given in Table 2.

4.2. Step 2: Final reshaping of obtuse triangles to acute triangles

As shown in Fig. 5a and c, obtuse triangles (dark regions) appear after *even bisections*. The number of obtuse triangles is half the number of total triangles as seen in the Figs. Since a pair of obtuse triangles exist, they can be reshaped to two acute triangles.

Fig. 6 illustrates the histogram of the sides' lengths of the final smallest triangles obtained by splitting the icosahedron by s=9 subdivisions (level 9). This figure clearly shows that: (1) the final

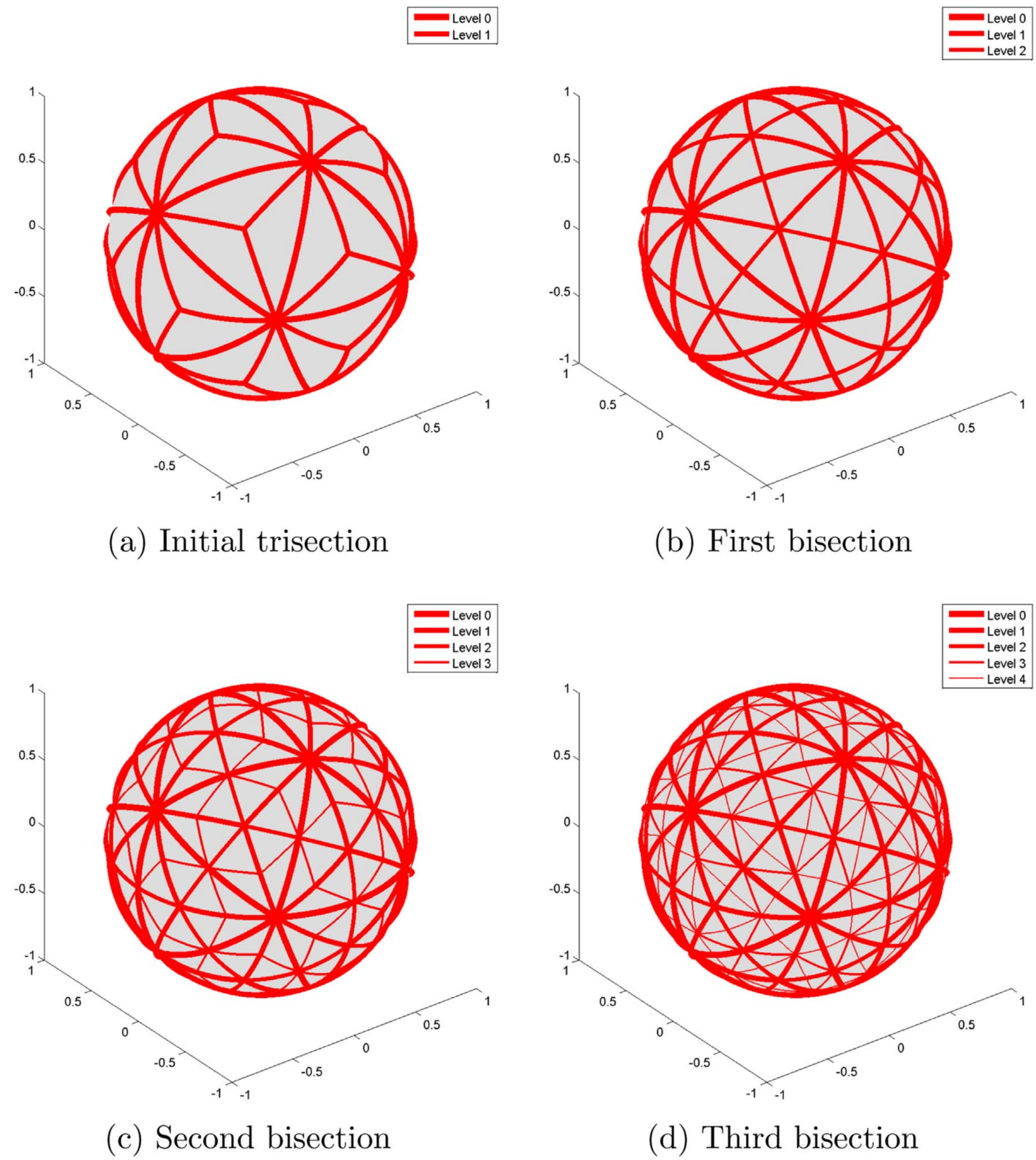


Fig. 4. Quasi-equal area subdivision algorithm - step 1. In this step, a icosahedron is first subdivided by the equal area trisection. Then, the trisection is followed by sequential equal area bisections. The relationship between the generated triangles and level is shown.

resulting triangle sides are bounded, meaning no degenerated triangles are obtained by recursive application of the splitting algorithm and (2) a smaller bound is obtained by final reshaping (because obtuse triangles disappear after reshaping).

Since the final reshaping step changes the area of each reshaped triangle, evaluation for the area preservation is performed with the method by Massey (2012). As shown in the left histogram of Fig. 7, all areas of the spherical triangles are within $\pm 5\%$ of the average area. The right histogram clearly shows that the spherical triangle vertex angles are bounded in a small range. Assuming that the area within $\pm 2.5\%$ of the mean area is preserved area, there are 75% of preserved area of spherical triangles in level 5 as illustrated in Fig. 8. At higher levels more spherical triangles can be area preserved.

Compared to the triangles by method of Teanby (2006), this algorithm has the advantage of having triangles of the quasi-equal area. In addition, it doesn't require the equalization process as in method of Massey (2012), which is quite computationally expensive. Equalization process is quite computationally expensive and time consuming.

4.3. Construction of quasi-uniform points

Mortari et al. (2011) suggested taking centers of the triangles as the quasi-uniform points since some of generated triangles are obtuse triangles. Since the quasi-equal area subdivision algorithm with even number of bisections provides all acute triangles, vertices of triangles can be considered to construct quasi-uniform points.

Visual results of the original algorithm (Mortari et al., 2011), Fig. 9a, and the proposed improvements, Figs. 9b and c, are provided.

Table 2
Number of vertices and faces of generated triangles in each level, along with the corresponding normalized time. Before the normalization, the time consumed level 1 grid is in the order of milliseconds on an Intel core i7 based machine.

Level	# of vertices	# of faces	subdivision time
0	12	20	n/a
1	32	60	1.0
2	62	120	1.5
3	122	240	2.2
4	242	480	3.1
5	482	960	5.5
6	962	1920	9.1
7	1922	3840	20
8	3842	7680	32
9	7682	15,360	87
10	15,362	30,720	125
11	30,722	61,440	508
12	61,442	122,880	961
13	122,882	245,760	8637

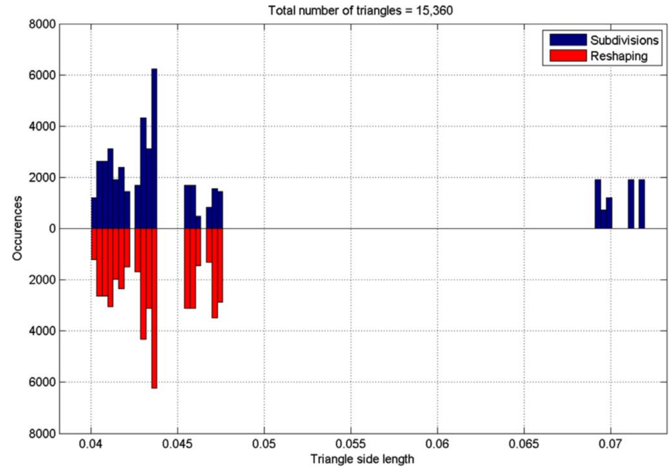
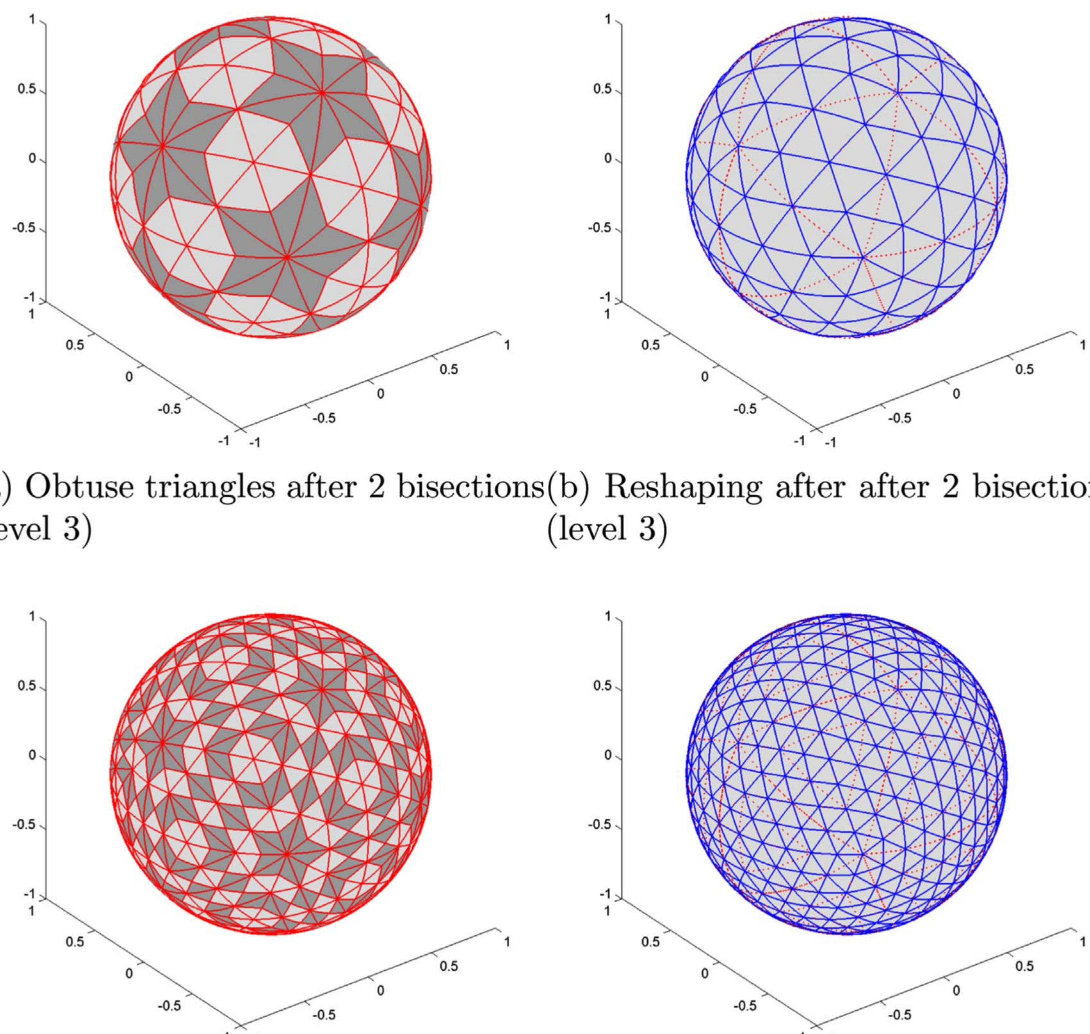


Fig. 6. Triangles sides length histograms (level 9) with and without reshaping. The upper shows triangle sides length histogram before reshaping, whereas the lower shows triangle sides length histogram after reshaping.



(a) Obtuse triangles after 2 bisections (level 3)
(b) Reshaping after after 2 bisections (level 3)
(c) Obtuse triangles after 4 bisections (level 5)
(d) Reshaping after after 4 bisections (level 5)

Fig. 5. Quasi-equal area subdivision algorithm - step 2. In this step, reshaping the final obtuse triangles after the final even number of bisections provides acute triangles. Since reshaping does not preserve the area of triangles, reshaping is applied one time only after all recursive divisions.

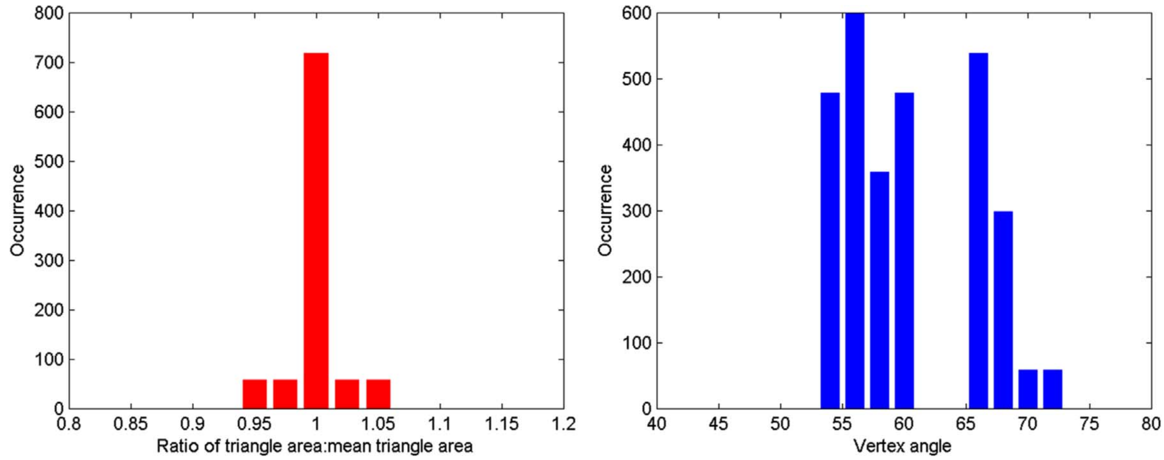


Fig. 7. Histogram of the ratio of spherical triangle area to mean spherical triangle area and vertex angles in a level 5 after subdividing from a level 0. The left figure indicates the area ratio, whereas the right figure indicates vertex angles of spherical triangle.

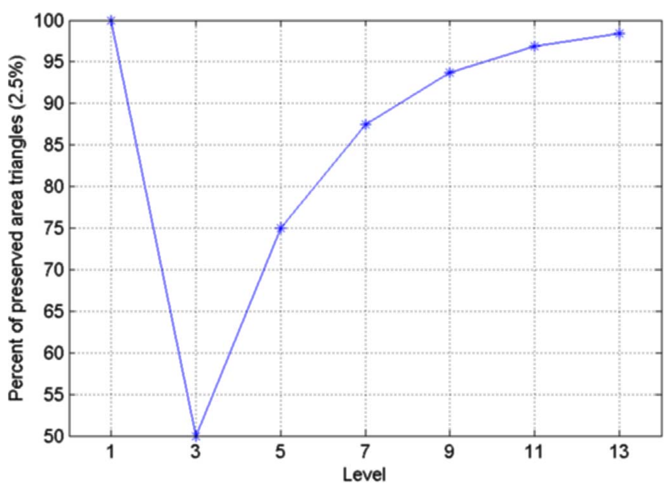


Fig. 8. Percent of preserved area spherical triangles (within 2.5% of the average area) in each level.

4.4. Smale's validation

The problem of uniform distribution of points on the sphere emerged from complexity theory in a paper by Shub and Smale (1994). Smale himself provided a mathematical tool to quantify the uniform distribution of points on a sphere. For any given distribution of points (unit-vectors), $\hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_N \in \mathbb{S}$, it is possible to evaluate the function V

$$V = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \log \frac{1}{\| \hat{\boldsymbol{x}}_i - \hat{\boldsymbol{x}}_j \|} \tag{11}$$

Let $V_{\min}$ be the minimum possible value of V. The problem asks for an algorithm that, for an assigned value of N, finds a sequence of points $\hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_N$ on the unit sphere such that

$$0 < V - V_{\min} \le c \log N \tag{12}$$

where c is a positive constant that depends only on the algorithm provided (Smale, 1998). Rakhmanov et al. (1994)) provide numerical evidence that their generalized spiral points algorithm supports Eq. (12) for $N \le 12,\ 000$, with c=114.

Smale provided (Smale, 1998) the following approximated (truncated) formula to evaluate $V_{\min}$

$$\widetilde{V}_{\min} = -\frac{1}{4} \log\left(\frac{4}{e}\right) N^2 - \frac{1}{4} N \log(N) + O(N) \tag{13}$$

where e is Euler's number. Eqs. (11)–(13) will be used to compare and quantify the uniformity of points distributions as generated by different algorithms.

Mortari et al. (2011) have introduced the following conjecture about the uniform distribution of points on a 2-dimensional surface. This conjecture is based on the recursive equal area subdivision of triangles just described.

Conjecture: *Using s recursive splits of an original spherical triangle a set of* 2^s *non-degenerating spherical triangles with identical areas is obtained. As* $s \to \infty$ *the centers of the final small spherical triangles identify a distribution of points satisfying Eqs.* (11) *and* (12) *for the original spherical triangle.*

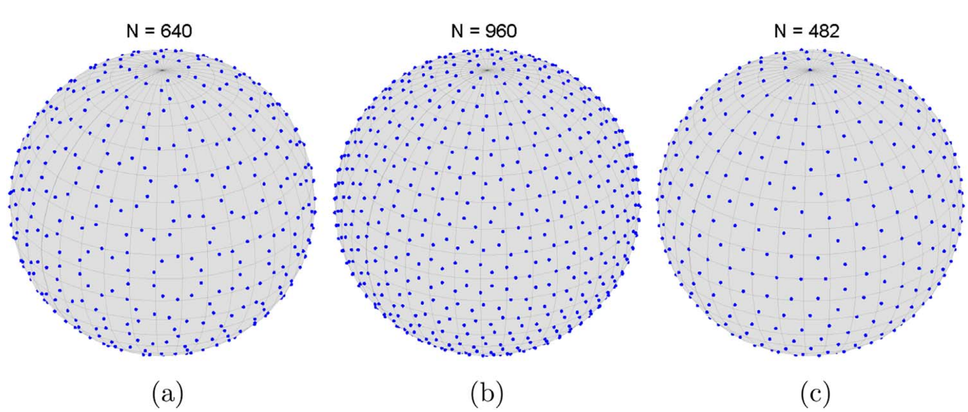


Fig. 9. Visual results of various algorithms (level 5). (a) Original equal area subdivision algorithm with centers of the triangles. (b) Quasi-equal area subdivision algorithm with centers of the triangles. (c) Quasi-equal area subdivision algorithm with vertices.

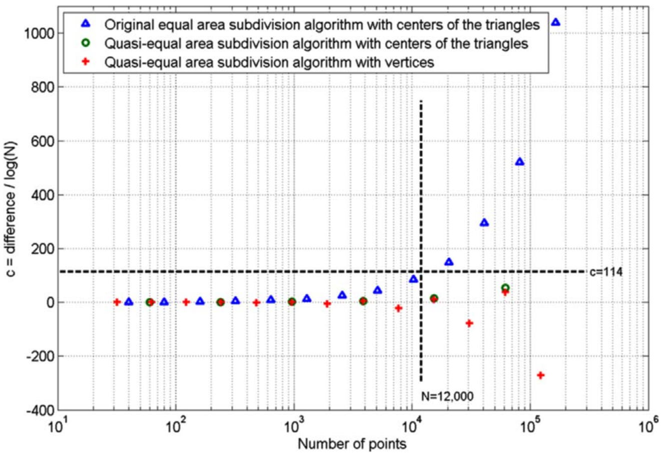


Fig. 10. Smale's validation results. Smale's validation measures uniformity by evaluating the difference between the value of V for the points obtained, as provided by Eq. (11), and the optimal value of $\widetilde{V}_{\min}$, as evaluated by Eq. (13).

If this conjecture is correct, then the creation of N asymptotically uniformly distributed points on a sphere depends on how the sphere is split in a spherical triangle.

The approach to measure uniformity of a set directions generated by an algorithm is to evaluate the difference between the value of V for the points obtained, as provided by Eq. (11), and the optimal value of $\widetilde{V}_{\min}$, as evaluated by Eq. (13). This approach is suggested by Smale (1998). The comparison results are shown in Fig. 10. Note that the algorithm using vertices with even number of bisections provides best performance since even number of bisections make generated triangles almost equal area triangles as illustrated in Fig. 5.

5. Applications to geographical data

Various kinds of geographical grid data sets are used in many fields such as sciences, economics, politics. For example, Gross National Product (GNP) and a worldwide population distribution map are used to estimate market demand for satellite (Chan et al., 2004). In the case of a global mission, the cost function for a constellation design is computed in globally distributed points (Park et al., 2005) and (Davis et al., 2013). Most gridded data sets are provided with a fixed step in latitude and longitude. Therefore, conventionally computed points (Teanby, 2006) are distributed with a fixed step in latitude and longitude as shown in Fig. 11a. Since this is certainly not a uniform distribution of points on the Earth, mainly due to the increase of point density at high latitude regions as illustrated in Fig. 11b, the need to convert these data into an "equivalent" distribution of points (with different weights) is needed. This will decrease to a small amount of data sets with appropriate values. To provide an equivalent resolution of the data, it is required to match the surface area of the bins for data to the area of a grid box in the gridded data. Consequently, computational burden is then reduced using "equivalent" uniformly distributed points.

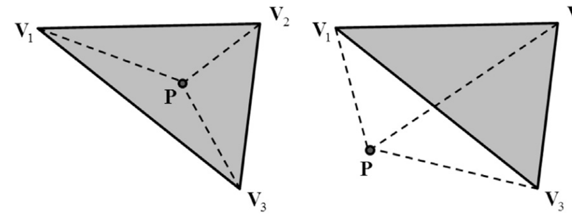


Fig. 12. Geometry of point inclusion and non-inclusion cases.

In the following subsections we discuss the method to check whether a specific point is included. Teanby (2006) suggested overlapped pentagonal and hexagonal bins. In this research, triangular and aperture cones methods have been used since these algorithms provide quasi-equivalent areas.

5.1. Binning check for geographical data

5.1.1. Triangular method

One method to convert this geographical data set into an equivalent quasi-uniform data set is using triangles. Many quasi-equal area triangles are provided by quasi-equal area spherical subdivision algorithms. Therefore, checking if the original grid data points are inside or outside a triangle can determine if points are included or not. Fig. 12 shows the geometry of inclusion and non-inclusion cases. In the case of inclusion (left), the sum of the areas of the three sub-triangles is equal to the area of the original triangle while the non-inclusion case (right) is experienced if the total area is greater than the area of the original triangle.

In order to perform the binning check, all uniformly distributed points are transformed to a topocentric-horizon coordinate system with the mid-point of the triangle as origin, and then projected to the surface as all points of the quadrilateral are transformed. The equation to check whether the point $\boldsymbol{P}_v$ is included is given by Lee and Mortari (2013a).

$$A(\boldsymbol{P}\boldsymbol{V}_1\boldsymbol{V}_2) + A(\boldsymbol{P}\boldsymbol{V}_2\boldsymbol{V}_3) + A(\boldsymbol{P}\boldsymbol{V}_3\boldsymbol{V}_1) = A(\boldsymbol{V}_1\boldsymbol{V}_2\boldsymbol{V}_3) \tag{14}$$

5.1.2. Aperture cones method

The triangular method can be applied only to algorithms using the center of triangles as illustrated in Fig. 13a. Therefore, a different method is required in order to use the algorithm with vertices which

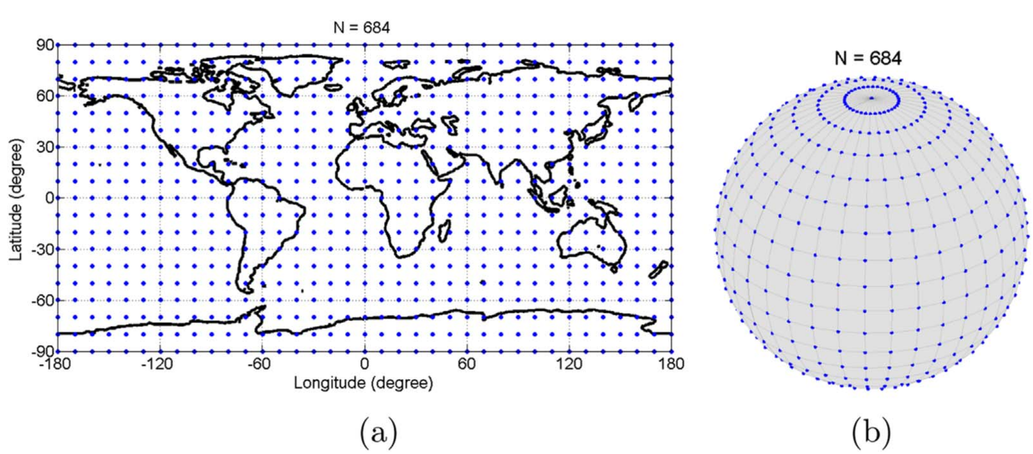
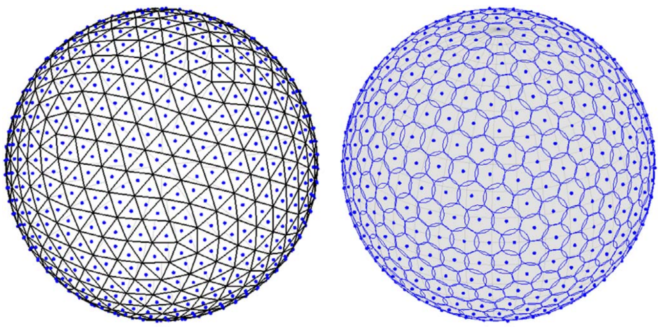


Fig. 11. Conventional scheme for distributing points (10° resolution).



(a) (b)

Fig. 13. Bins with quasi-uniform points. (a) Bins are equal area triangles in the triangular method (b) Bins are aperture of which centers are quasi-uniform points in the aperture cones method.

provide best performance. This is the aperture cones method which requires only distributed points as shown in Fig. 13b. Considering N directions to distributed points from the origin with an angular aperture, N aperture cones are assigned. Checking if original grid points lie inside or outside the aperture can be accomplished by finding the angle between two direction vectors. The cosine of the angle between the direction of distributed point and the direction of original grid point is found by dividing the scalar product of the vectors by the product of their magnitudes. The following two subsections with two examples for geographical data application demonstrate binning methods together.

5.2. Example 1: Building uniform sampled data with regional grid data

The first example is a case with regional grid data. The Nitrogen Fertilizer Application data set of the Global Fertilizer and Manure, Version 1 Data Collection represents the amount of nitrogen fertilizer nutrients applied to croplands. The data were compiled by Potter et al. (2010) and are distributed by the Columbia University Center for International Earth Science Information Network (CIESIN). Data are provided at 1° resolution in fixed latitude by longitude coordinates.

In this example the quasi-area subdivision algorithm with centers of triangles has been applied, and the triangular method has been used for binning data. The surface area of triangle is matched to the area of the grid box at the lowest latitude in the original data. Compare to Fig. 14a, the final data set is maintaining spatial resolution over the globe as shown in Fig. 14b.

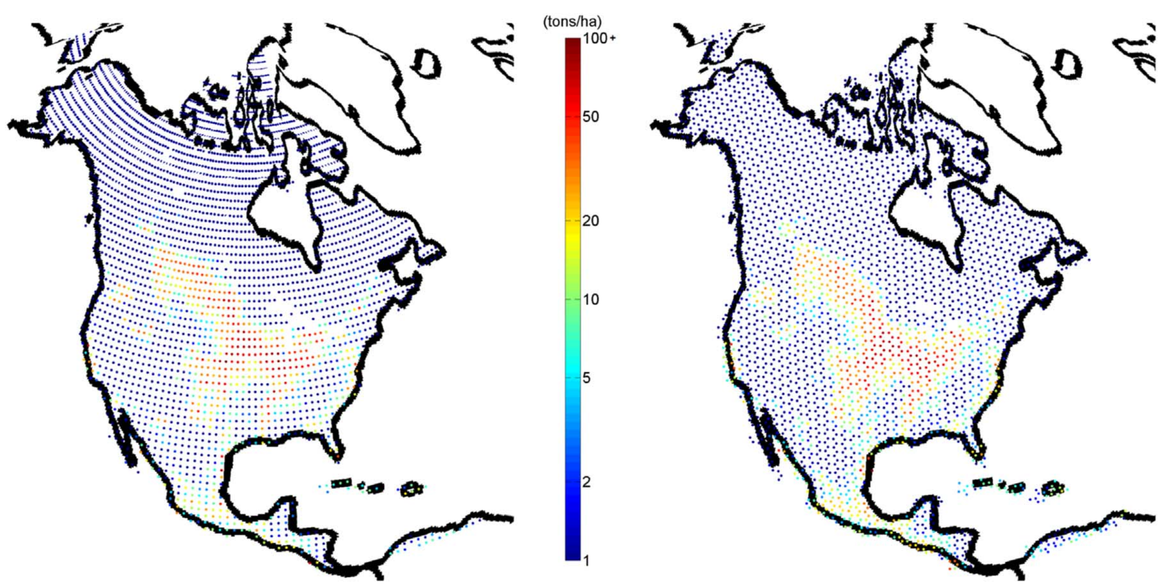
5.3. Example 2: Building uniform sampled data with global grid data

The second example is a case with global grid data. Gridded Population of the World, Version 3 (GPWv3), Future Estimates consists of projections of human population for the year 2015 by 1° grid cells. A proportional allocation gridding algorithm, utilizing more than 300,000 national and sub-national administrative units, is used to assign population values to grid cells. The population density grids are derived by dividing the population count grids by the land area grid and represent persons per square kilometer (Center for International Earth Science Information Network (CIESIN)/Columbia University or International Earth Science Information Network (CIESIN)/Columbia University, and Centro Internacional de Agricultura Tropical (CIAT), 2005).

The quasi-area subdivision algorithm with vertices has been used in this example. Therefore the aperture cones method has been used for binning data. The aperture area is approximately equal to the area of the grid box at the equator in the original data. Finally latitude distorted geographical points in Fig. 15a are converted to uniform points across the sphere as illustrated in Fig. 15b.

6. Conclusions

This paper provides a quasi-equal area subdivision algorithm based on equal area spherical subdivisions to obtain uniform distribution of points on a sphere. The algorithm adopts the theory of the original equal area subdivision algorithm, which performs subsequent bisections of spherical triangles. The whole sphere which is made of N equal area spherical triangles can be obtained by subsequent subdivisions. As the number of divisions increases, the center of these spherical N triangles provides a good approximation to the uniform distribution of N points on a 2-dimensional surface. The main feature of the proposed



(a) (b)

Fig. 14. Building uniform sampled data with regional grid data for nitrogen fertilizer application. (a) The nitrogen fertilizer application data set has 1 degree of resolution (1358 points). Effects of latitude distortion increase spatial resolution. (b) Quasi-equal area subdivision algorithm (level 11) with centers of triangles provides a constant spatial resolution data set (983 points).

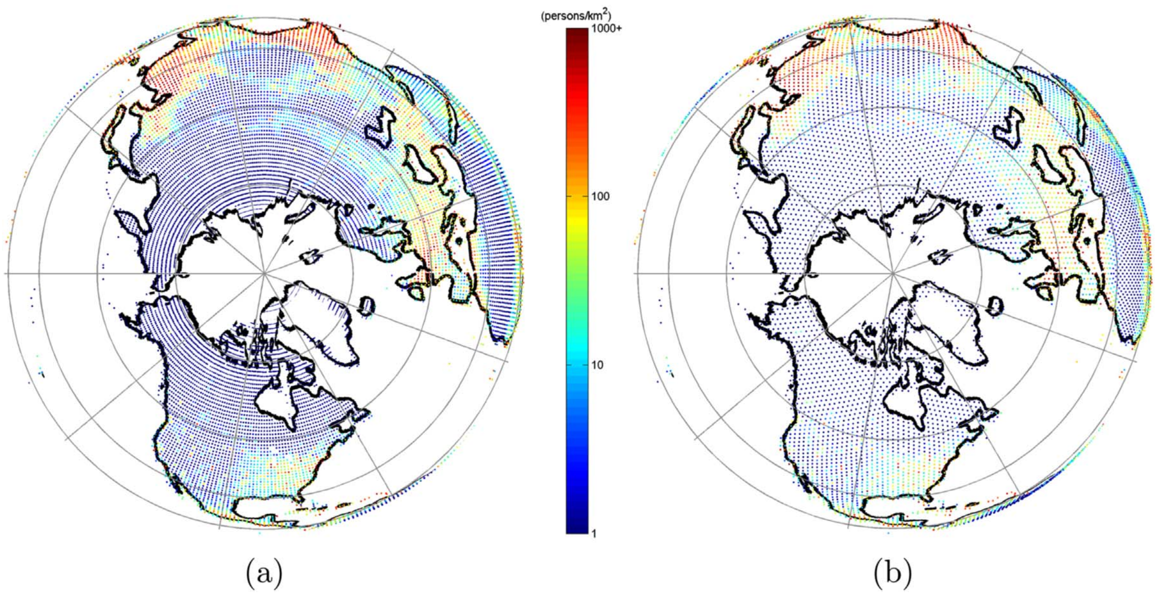


Fig. 15. Building uniform sampled data with global grid data for population estimates. (a) The population estimates data set has 1 degrees resolution (15,648 points). Effects of latitude distortion increase spatial resolution. (b) Quasi-equal area subdivision algorithm (level 11) with vertices maintains constant spatial resolution data set over the globe (9566 points).

algorithm is to share common vertices between adjacent triangles. This is accomplished by the initial trisection since it symmetrically deploys generated triangles. Therefore, reshaping can be applied to triangles, and vertices can be used for uniform points. It has been found that the proposed quasi-equal area subdivision algorithm provides good performance with validation.

After generating uniformly distributed points, two methods for binning data have been proposed. These binning check methods have been used successfully in two examples which demonstrate geographical data applications. These examples also demonstrate that suggested algorithms maintain a constant spatial resolution over the globe, which is required in most applications where gridding bias is to be avoided.

Appendix A. Supplementary data

Supplementary data associated with this article can be found in the online version at http://dx.doi.org/10.1016/j.cageo.2017.03.012.

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