Computers & Geosciences 88 (2016) 152–162

Contents lists available at ScienceDirect

Computers & Geosciences

journal homepage: www.elsevier.com/locate/cageo

Case study

Non-singular expressions for the spherical harmonic synthesis of gravitational curvatures in a local north-oriented reference frame

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ARTICLE INFO

Article history:
Received 30 July 2015
Received in revised form
7 December 2015
Accepted 13 December 2015
Available online 17 December 2015

Keywords:
Associated Legendre's function
Global gravitational model
Gravitational curvature
Harmonic synthesis
Spherical harmonic expansion

ABSTRACT

Third-order gradients of the gravitational potential (gravitational curvatures) have already found some applications in geosciences. Observability of these parameters, describing the Earth's gravitational field in a more complex way than any other currently available gravitational parameter, such as gravitational acceleration (first-order gradient) or gravitational (second-order) gradient, is currently discussed by physicists. Moreover, first designs of observational devices (sensors) have already been proposed. The spherical harmonic analysis and synthesis are the common tools used by geoscientists to study spectral properties of various functionals of the Earth's gravitational potential. However, the conventional spherical harmonic expansions of the gravitational curvatures in the local north-oriented reference frame have rather complicated forms that depend on the first-, second- and third-order derivatives of the associated Legendre functions. Moreover, some of these expansions also contain singular terms at the poles. In this paper, the conventional series are transformed to new simpler and non-singular forms based on relations between the associated Legendre functions and their derivatives. Numerical experiments demonstrate the applicability and correctness of the new expressions.

1. Introduction

Currently available ground, marine, airborne and satellite sensors for Earth's gravity field mapping allow for gravity or gravitational acceleration measurements. The relative accuracy of ground gravity measurements reached the ppb level. However, the accuracy of measurements performed at moving platforms is significantly worse. Moreover, the full signal cannot usually be recovered due to extensive data filtering in order to reduce the observation noise. If the full gravitational acceleration vector is measured (vector gravimetry), we can recover all components of the first-order gravitational tensor that consists of three gravitational gradients in a specific coordinate frame. By combining gravitational acceleration measurements the second-order gravitational tensor can be derived. This is basically the observational principle of the gravity-dedicated satellite mission GOCE (Gravity field and steady-state Ocean Circulation Explorer), see, e.g., ESA (1999) and Rummel (2010). Gravitational observables and/or derived pseudo-observables have widely been exploited by geoscientists for the purpose of the Earth's gravitational field modelling, interpretation and spectral analyses.

In recent years, new sensors for observing a third-order gravitational tensor have been proposed. The Russian project Dulkyn (www.dulkyn.ru) aims at developing a system that would eventually observe third-order directional derivatives of the gravitational potential (in case of the Earth's gravitational field also called shortly geopotential) together with their temporal variations (Balakin et al., 1997). Recently, Rosi et al. (2015) performed first measurements of the third-order vertical derivatives of the geopotential at the Earth's surface. The gravity-dedicated satellite mission called OPTIMA (OPTical Interferometry for global MAss change detection from space) designed to measure the third-order derivatives of the geopotential was proposed by Brieden et al. (2010). Motivated by higher sensitivity of the third-order derivatives of the geopotential to short-wavelength structures of the Earth's gravitational field (Jacoby and Smilde, 2009), their exploitation has repeatedly been suggested for geophysical exploration purposes, see, e.g., Troshkov and Shalaev (1968), Smith et al. (1998), Fedi and Florio (2001), Thurston et al. (2002), Abdelrahman et al. (2003), Hafez et al. (2006), Pajot et al. (2008), Veryaskin and McRae (2008), Beiki (2010) and Eppelbaum (2011).

These ongoing efforts opened a new chapter in the area of observability of gravitation (gravity stripped of centrifugal acceleration). In geodesy, the third-order derivatives of the geopotential have been discussed in various contexts. Already Moritz (1967) investigated parameters of the Earth's gravitational field up to the third-order gravitational tensor and showed that gravitational tensors of the order three and higher were independent of sensors'

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http://dx.doi.org/10.1016/j.cageo.2015.12.011
0098-3004/

orientation. Thus, any instrument capable of observing such quantities would provide a pure gravitational signal. Rummel (1986), Rummel et al. (1993), Koop (1993) and Albertella et al. (2000) used the third-order gravitational tensor for the error analysis of gradiometric observations. Ardalan and Grafarend (2001) expanded the normal gravitational potential (generated by a homogeneous geocentric biaxial ellipsoid) into the Taylor series up to the third-order geopotential derivatives in analyzing Bruns's formula. Grafarend (1997) also derived a functional relationship between curvature and torsion of a plumb-line to second- and third-order derivatives of the geopotential. Casotto and Fantino (2009) derived expressions for gravitational tensors up to the third order in local and global reference frames by tensor analysis. Most recently, Šprlák and Novák (2015) studied functional relationships between third-order geopotential derivatives and gravitating mass density distribution, anomalous gravitational acceleration and the geopotential.

Spectral properties of the gravitational field based on the third-order derivatives of the geopotential have also been studied. Cunningham (1970) derived spherical harmonic series for gravitational tensors of an arbitrary order in a geocentric reference frame. This study was extended by Metris et al. (1999) and Petrovskaya and Vershkov (2010). Computational aspects of the harmonic synthesis up to the third-order derivatives of Legendre's functions were discussed by Fantino and Casotto (2009) and Fukushima (2012, 2013). Non-singular expressions for a geomagnetic vector and gradient tensor fields were also studied by Du et al. (2015). Expressions for both the spherical harmonic analysis and synthesis use associated Legendre's functions of the first kind. Expressions for the third-order derivatives of the geopotential in spherical coordinates include respective derivatives of the associated Legendre functions. Moreover, they include terms dependent on latitude which are singular at both poles.

In this paper recursive expressions for computing values of the third-order derivatives of the associated Legendre functions are given. Formulas for spherical harmonic synthesis are modified to avoid numerical instabilities including singularities at the poles. A simple analytical structure of the new expressions is particularly suitable for deriving geopotential coefficients from eventually available observables as well as for studying spectral properties of the Earth's gravitational field based on the third-order gradients of the geopotential. Conventional expansions for the gravitational curvatures in a local north-oriented reference frame (LNOF) are transformed on the basis of relations given by Ilk (1983), Petrovskaya and Vershkov (2006) and Eshagh (2008, 2010).

The paper is organized as follows: in Section 2 we formulate the problem, define differential operators and provide conventional and new non-singular expressions for the series representation of the components of the third-order gravitational tensor in LNOF; in Section 3 we provide underlying expressions for derivatives of the associated Legendre functions and outline derivation of the new non-singular expressions; Section 4 contains numerical results obtained through computer realizations of the new formulas that demonstrate their correctness and functionality at the poles; finally, contributions of the paper are summarized in Conclusions. We also provide a MATLAB based program for potential users.

2. Formulation of the problem

In the following we define the Earth's disturbing potential T as a difference of the Earth's gravitational potential reduced for the gravitational potential of the geocentric homogeneous biaxial ellipsoid (such as GRS80, Moritz, 2000). Outside the Earth's masses (and the reference ellipsoid), the disturbing potential T is a harmonic function as it satisfies the Laplace–Poisson differential equation. Thus, it can be represented by a spherical harmonic series of the form, e.g., Heiskanen and Moritz (1967, Section 2–14):

$$T(r,\varphi,\lambda) = \frac{GM}{a}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^{n+1}\sum_{m=0}^{n}\bar{P}_{n,m}(\sin\varphi)\,(\bar{C}_{n,m}\cos m\lambda + \bar{S}_{n,m}\sin m\lambda). \tag{1}$$

In this equation, GM represents the geocentric gravitational constant, a is the radius of a mean geocentric sphere approximating the Earth, $\bar{P}_{n,m}$ is the fully normalized associated Legendre function of the first kind of degree n and order m, and $\bar{C}_{n,m}$ and $\bar{S}_{n,m}$ are respective fully normalized spherical harmonic coefficients. The disturbing potential T is a function of three coordinates (neglecting its temporal variability) that define its location in 3-D space: in our particular case they include geocentric radius r, spherical latitude φ and longitude λ. As available observations (and numerical limitations) allow for determination only of a finite number of the spherical harmonic coefficients, the series is truncated at some maximum degree N (currently at the level of $\approx$ 2000).

The third-order gravitational tensor is represented by 27 gravitational third-order geopotential gradients (gravitational curvatures) but only 10 of them are distinct from each other because of continuity of the Earth's gravitational field. In this paper, we will consider only gravitational curvatures referred to LNOF. Such a reference frame is defined by an origin in the point of interest and by a right-handed orthogonal basis with the following orientation of axes: the x-axis points to the North, the y-axis points to the West and the z-axis is directed radially outward. Each of these ten gravitational curvatures is defined by one differential operator. Such differential operators in terms of the spherical geocentric coordinates read as follows (Tóth, 2005; Casotto and Fantino, 2009):

$$\mathcal{D}^{xxx} = -\frac{1}{r^2}\left(\frac{2}{r}\frac{\partial}{\partial\varphi} - 3\frac{\partial^2}{\partial r\partial\varphi} - \frac{1}{r}\frac{\partial^3}{\partial\varphi^3}\right), \tag{2}$$

$$\mathcal{D}^{xxy} = -\frac{1}{r^2\cos\varphi}\left(\frac{2\tan^2\varphi}{r}\frac{\partial}{\partial\lambda} + \frac{\partial^2}{\partial r\partial\lambda} + \frac{2\tan\varphi}{r}\frac{\partial^2}{\partial\varphi\partial\lambda} + \frac{1}{r}\frac{\partial^3}{\partial\varphi^2\partial\lambda}\right), \tag{3}$$

$$\mathcal{D}^{xxz} = -\frac{1}{r}\left(\frac{1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial r^2} + \frac{2}{r^2}\frac{\partial^2}{\partial\varphi^2} - \frac{1}{r}\frac{\partial^3}{\partial r\partial\varphi^2}\right), \tag{4}$$

$$\mathcal{D}^{xyy} = -\frac{1}{r^2}\left(\frac{1}{r\cos^2\varphi}\frac{\partial}{\partial\varphi} - \frac{\partial^2}{\partial r\partial\varphi} + \frac{\tan\varphi}{r}\frac{\partial^2}{\partial\varphi^2} - \frac{2\tan\varphi}{r\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2} - \frac{1}{r\cos^2\varphi}\frac{\partial^3}{\partial\varphi\partial\lambda^2}\right), \tag{5}$$

$$\mathcal{D}^{xyz} = \frac{1}{r^2\cos\varphi}\left(\frac{2\tan\varphi}{r}\frac{\partial}{\partial\lambda} - \tan\varphi\frac{\partial^2}{\partial r\partial\lambda} + \frac{2}{r}\frac{\partial^2}{\partial\varphi\partial\lambda} - \frac{\partial^3}{\partial r\partial\varphi\partial\lambda}\right), \tag{6}$$

$$\mathcal{D}^{xzz} = \frac{1}{r}\left(\frac{2}{r^2}\frac{\partial}{\partial\varphi} - \frac{2}{r}\frac{\partial^2}{\partial r\partial\varphi} + \frac{\partial^3}{\partial r^2\partial\varphi}\right), \tag{7}$$

$$\mathcal{D}^{yyy} = \frac{1}{r^2\cos\varphi}\left(\frac{2}{r\cos^2\varphi}\frac{\partial}{\partial\lambda} - 3\frac{\partial^2}{\partial r\partial\lambda} + \frac{3\tan\varphi}{r}\frac{\partial^2}{\partial\varphi\partial\lambda} - \frac{1}{r\cos^2\varphi}\frac{\partial^3}{\partial\lambda^3}\right), \tag{8}$$

$$\mathcal{D}^{yyz} = -\frac{1}{r}\left(\frac{1}{r}\frac{\partial}{\partial r} - \frac{2\tan\varphi}{r^2}\frac{\partial}{\partial\varphi} - \frac{\partial^2}{\partial r^2} + \frac{\tan\varphi}{r}\frac{\partial^2}{\partial r\partial\varphi} + \frac{2}{r^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2} - \frac{1}{r\cos^2\varphi}\frac{\partial^3}{\partial r\partial\lambda^2}\right), \tag{9}$$

$$\mathcal{D}^{yzz} = -\frac{1}{r\cos\varphi}\left(\frac{2}{r^2}\frac{\partial}{\partial\lambda} - \frac{2}{r}\frac{\partial^2}{\partial r\partial\lambda} + \frac{\partial^3}{\partial r^2\partial\lambda}\right), \tag{10}$$

$$\mathcal{D}^{zzz} = \frac{\partial^3}{\partial r^3}. \tag{11}$$

The differential operators of Eqs. (2)–(11) can be applied to the disturbing potential of Eq. (1). As the equations become lengthy, we introduce a simplified notation and some substitutions. First of all, we skip from now on the coordinate parameters r, φ and λ associated with the gravitational curvatures $T_{\mu\nu\tau}$; $\mu, \nu, \tau \in \{x, y, z\}$. We also omit the parameter $\sin\varphi$ of the associated Legendre function $\bar{P}_{n,m}$. Finally, we introduce the degree-dependent scaling parameter

$$\kappa_n(r) = \frac{GM}{a^4}\left(\frac{a}{r}\right)^{n+4},$$

that is responsible for the physical dimension of the gravitational curvature, i.e., $\mathrm{m}^{-1}\,\mathrm{s}^{-2}$. Then the truncated spherical harmonic series for the third-order gravitational gradients, i.e., the conventional expressions, have the following forms:

$$T_{xxx} = -\sum_{n=2}^{N}\kappa_n(r)\sum_{m=0}^{n}\left[(3n+5)\frac{\mathrm{d}\bar{P}_{n,m}}{\mathrm{d}\varphi} - \frac{\mathrm{d}^3\bar{P}_{n,m}}{\mathrm{d}\varphi^3}\right](\bar{C}_{n,m}\cos m\lambda + \bar{S}_{n,m}\sin m\lambda), \tag{12}$$

$$T_{xxy} = \frac{1}{\cos\varphi}\sum_{n=2}^{N}\kappa_n(r)\sum_{m=0}^{n}\left[(2\tan^2\varphi - n - 1)\bar{P}_{n,m} + 2\tan\varphi\frac{\mathrm{d}\bar{P}_{n,m}}{\mathrm{d}\varphi} + \frac{\mathrm{d}^2\bar{P}_{n,m}}{\mathrm{d}\varphi^2}\right]m(\bar{S}_{n,m}\cos m\lambda - \bar{C}_{n,m}\sin m\lambda), \tag{13}$$

$$T_{xxz} = \sum_{n=2}^{N}\kappa_n(r)(n+3)\sum_{m=0}^{n}\left[(n+1)\bar{P}_{n,m} - \frac{\mathrm{d}^2\bar{P}_{n,m}}{\mathrm{d}\varphi^2}\right](\bar{C}_{n,m}\cos m\lambda + \bar{S}_{n,m}\sin m\lambda), \tag{14}$$

$$T_{xyy} = -\sum_{n=2}^{N}\kappa_n(r)\sum_{m=0}^{n}\left[\frac{2\tan\varphi}{\cos^2\varphi}m^2\,\bar{P}_{n,m} + \left(\frac{1+m^2}{\cos^2\varphi} + n + 1\right)\frac{\mathrm{d}\bar{P}_{n,m}}{\mathrm{d}\varphi} + \tan\varphi\frac{\mathrm{d}^2\bar{P}_{n,m}}{\mathrm{d}\varphi^2}\right](\bar{C}_{n,m}\cos m\lambda + \bar{S}_{n,m}\sin m\lambda), \tag{15}$$

$$T_{xyz} = \frac{1}{\cos\varphi}\sum_{n=2}^{N}\kappa_n(r)(n+3)\sum_{m=0}^{n}\left(\tan\varphi\bar{P}_{n,m} + \frac{\mathrm{d}\bar{P}_{n,m}}{\mathrm{d}\varphi}\right)\times m(\bar{S}_{n,m}\cos m\lambda - \bar{C}_{n,m}\sin m\lambda), \tag{16}$$

$$T_{xzz} = \sum_{n=2}^{N}\kappa_n(r)(n+2)(n+3)\sum_{m=0}^{n}\frac{\mathrm{d}\bar{P}_{n,m}}{\mathrm{d}\varphi}(\bar{C}_{n,m}\cos m\lambda + \bar{S}_{n,m}\sin m\lambda), \tag{17}$$

$$T_{yyy} = \frac{1}{\cos\varphi}\sum_{n=2}^{N}\kappa_n(r)\sum_{m=0}^{n}\left\{\left[\frac{m^2+2}{\cos^2\varphi} + 3(n+1)\right]\bar{P}_{n,m} + 3\tan\varphi\frac{\mathrm{d}\bar{P}_{n,m}}{\mathrm{d}\varphi}\right\}m(\bar{S}_{n,m}\cos m\lambda - \bar{C}_{n,m}\sin m\lambda), \tag{18}$$

$$T_{yyz} = \sum_{n=2}^{N}\kappa_n(r)(n+3)\sum_{m=0}^{n}\left[\left(\frac{m^2}{\cos^2\varphi} + n + 1\right)\bar{P}_{n,m} + \tan\varphi\frac{\mathrm{d}\bar{P}_{n,m}}{\mathrm{d}\varphi}\right]\times(\bar{C}_{n,m}\cos m\lambda + \bar{S}_{n,m}\sin m\lambda), \tag{19}$$

$$T_{yzz} = -\frac{1}{\cos\varphi}\sum_{n=2}^{N}\kappa_n(r)(n+2)(n+3)\sum_{m=0}^{n}\bar{P}_{n,m}m(\bar{S}_{n,m}\cos m\lambda - \bar{C}_{n,m}\sin m\lambda), \tag{20}$$

$$T_{zzz} = -\sum_{n=2}^{N}\kappa_n(r)(n+1)(n+2)(n+3)\sum_{m=0}^{n}\bar{P}_{n,m}(\bar{C}_{n,m}\cos m\lambda + \bar{S}_{n,m}\sin m\lambda). \tag{21}$$

Equations (12)–(19) contain the first-, second- and third-order latitudinal derivatives of the associated Legendre functions. In addition, Eqs. (13), (15), (16), (18), (19) and (20) also contain terms which are singular at the poles. Note that cosine of latitude in the denominator of the spherical harmonic series does not always cause singularity, see Fukushima (2012); however, in case of the above equations singularities were confirmed by numerical experiments. In the next section, new non-singular expressions are derived which do not contain any derivatives of the associated Legendre functions. These new expressions have the following forms:

$$T_{xxx} = -\sum_{n=2}^{N}\kappa_n(r)\sum_{m=0}^{n}(a_{n,m}\bar{P}_{n,m-3} + b_{n,m}\bar{P}_{n,m-1} + c_{n,m}\bar{P}_{n,m+1} + d_{n,m}\bar{P}_{n,m+3})(\bar{C}_{n,m}\cos m\lambda + \bar{S}_{n,m}\sin m\lambda), \tag{22}$$

$$T_{xxy} = \sum_{n=2}^{N}\kappa_n(r)\sum_{m=1}^{n}(e_{n,m}\bar{P}_{n-1,m-3} + f_{n,m}\bar{P}_{n-1,m-1} + g_{n,m}\bar{P}_{n-1,m+1} + h_{n,m}\bar{P}_{n-1,m+3})\,m(\bar{S}_{n,m}\cos m\lambda - \bar{C}_{n,m}\sin m\lambda), \tag{23}$$

$$T_{xxz} = \sum_{n=2}^{N}\kappa_n(r)(n+3)\sum_{m=0}^{n}(k_{n,m}\bar{P}_{n,m-2} + l_{n,m}\bar{P}_{n,m} + o_{n,m}\bar{P}_{n,m+2})\times(\bar{C}_{n,m}\cos m\lambda + \bar{S}_{n,m}\sin m\lambda), \tag{24}$$

$$T_{xyy} = \sum_{n=2}^{N}\kappa_n(r)\sum_{m=0}^{n}(a_{n,m}\bar{P}_{n,m-3} + q_{n,m}\bar{P}_{n,m-1} + r_{n,m}\bar{P}_{n,m+1} + d_{n,m}\bar{P}_{n,m+3})\times(\bar{C}_{n,m}\cos m\lambda + \bar{S}_{n,m}\sin m\lambda), \tag{25}$$

$$T_{xyz} = \sum_{n=2}^{N}\kappa_n(r)(n+3)\sum_{m=1}^{n}(s_{n,m}\bar{P}_{n-1,m-2} + v_{n,m}\bar{P}_{n-1,m} + w_{n,m}\bar{P}_{n-1,m+2})(\bar{S}_{n,m}\cos m\lambda - \bar{C}_{n,m}\sin m\lambda), \tag{26}$$

$$T_{xzz} = \sum_{n=2}^{N}\kappa_n(r)(n+2)(n+3)\sum_{m-1}^{n}(\beta_{n,m}\bar{P}_{n,m-1} + \gamma_{n,m}\bar{P}_{n,m+1})\times(\bar{C}_{n,m}\cos m\lambda + \bar{S}_{n,m}\sin m\lambda), \tag{27}$$

$$T_{yyy} = \sum_{n=2}^{N} \kappa_n(r) \sum_{m=1}^{n} (- e_{n,m}\bar{P}_{n-1,m-3} + \epsilon_{n,m}\bar{P}_{n-1,m-1} + \mu_{n,m}\bar{P}_{n-1,m+1} - h_{n,m}\bar{P}_{n-1,m+3})\, m(\bar{S}_{n,m}\cos m\lambda - \bar{C}_{n,m}\sin m\lambda), \tag{28}$$

$$T_{yyz} = -\sum_{n=2}^{N} \kappa_n(r)(n+3) \sum_{m=0}^{n} (k_{n,m}\bar{P}_{n,m-2} + \nu_{n,m}\bar{P}_{n,m} + o_{n,m}\bar{P}_{n,m+2}) \times (\bar{C}_{n,m}\cos m\lambda + \bar{S}_{n,m}\sin m\lambda), \tag{29}$$

$$T_{yzz} = -\sum_{n=2}^{N} \kappa_n(r)(n+2)(n+3) \sum_{m=1}^{n} (\tau_{n,m}\bar{P}_{n-1,m-1} + \chi_{n,m}\bar{P}_{n-1,m+1}) \times m(\bar{S}_{n,m}\cos m\lambda - \bar{C}_{n,m}\sin m\lambda). \tag{30}$$

The numerical coefficients $a_{n,m} - \chi_{n,m}$ of the above series expansions can be found in Appendix A. These numerical coefficients are constants in contrast to the original latitude-dependent coefficients in the conventional formulas of Eqs. (12)–(20). Note that the expression for T_{zzz} remains the same as it contains neither singularities nor derivatives of the associated Legendre function; moreover, all expressions are also non-orthogonal (except for T_{zzz}).

The new equations can be used for summations starting with $n=0$. Only expressions for the components with the odd (first or third) order derivative by y are summed up from $n=1$, see Eqs. (23), (26), (28) and (30). The second summation then starts from $m=1$. The same expressions contain the associated Legendre functions of the degree $n-1$. Besides that, the longitude-dependent part of the equations has a form which differs from the longitude-dependent terms in the equations for the components with the zero- or second-order derivatives by y.

Some similarities between Eqs. (21) and (30) may be observed. For example, equations for the components with three horizontal derivatives (with respect to x and/or y) contain four terms with the associated Legendre function, see Eqs. (22), (23), (25) and (28). The components with two horizontal derivatives contain three terms with the associated Legendre function, see Eqs. (24), (26) and (29). The components with one horizontal derivative contain two terms with the associated Legendre function, see Eqs. (27) and (30). Finally, the component T_{zzz} contains only one term with the associated Legendre function, see Eq. (21). There are also some similarities regarding numerical coefficients. For example, Eqs. (22) and (25) contain two identical coefficients. There are also two identical coefficients in Eqs. (23) and (28), and additional two in Eqs. (24) and (29).

At the end of this section, formulas for the spherical harmonic synthesis of the third-order geopotential gradients at the poles are stated. These equations were derived using an approach similar to that of Bosch (2000). As may be seen, the summation over m and the Legendre functions of the first kind are no longer present in the following equations:

$$T_{xxx} = -\sum_{n=2}^{N} \kappa_n(r)\sqrt{2n+1}\Big[a_{n,3}(\bar{C}_{n,3}\cos 3\lambda + \bar{S}_{n,3}\sin 3\lambda) + b_{n,1}(\bar{C}_{n,1}\cos\lambda + \bar{S}_{n,1}\sin\lambda)\Big]p_0, \tag{31}$$

$$T_{xxy} = \sum_{n=2}^{N} \kappa_n(r)\sqrt{2n-1}\Big[3e_{n,3}(\bar{S}_{n,3}\cos 3\lambda - \bar{C}_{n,3}\sin 3\lambda) + f_{n,1}(\bar{S}_{n,1}\cos\lambda - \bar{C}_{n,1}\sin\lambda)\Big]p_1, \tag{32}$$

$$T_{xxz} = \sum_{n=2}^{N} \kappa_n(r)\sqrt{2n+1}\,(n+3) \times \Big[k_{n,2}(\bar{C}_{n,2}\cos 2\lambda + \bar{S}_{n,2}\sin 2\lambda) + l_{n,0}\bar{C}_{n,0}\Big]p_0, \tag{33}$$

$$T_{xyy} = \sum_{n=2}^{N} \kappa_n(r)\sqrt{2n+1}\Big[a_{n,3}(\bar{C}_{n,3}\cos 3\lambda + \bar{S}_{n,3}\sin 3\lambda) + q_{n,1}(\bar{C}_{n,1}\cos\lambda + \bar{S}_{n,1}\sin\lambda)\Big]p_0, \tag{34}$$

$$T_{xyz} = \sum_{n=2}^{N} \kappa_n(r)\sqrt{2n-1}\,(n+3)\,s_{n,2}(\bar{C}_{n,2}\cos 2\lambda + \bar{S}_{n,2}\sin 2\lambda)p_1, \tag{35}$$

$$T_{xzz} = \sum_{n=2}^{N} \kappa_n(r)\sqrt{2n+1}\,(n+2)(n+3)\,\beta_{n,1}(\bar{C}_{n,1}\cos\lambda + \bar{S}_{n,1}\sin\lambda)p_0, \tag{36}$$

$$T_{yyy} = \sum_{n=2}^{N} \kappa_n(r)\sqrt{2n-1}\Big[-3e_{n,3}(\bar{S}_{n,3}\cos 3\lambda - \bar{C}_{n,3}\sin 3\lambda) + \epsilon_{n,1}(\bar{S}_{n,1}\cos\lambda - \bar{C}_{n,1}\sin\lambda)\Big]p_1, \tag{37}$$

$$T_{yyz} = -\sum_{n=2}^{N} \kappa_n(r)\sqrt{2n+1}\,(n+3) \times \Big[k_{n,2}(\bar{C}_{n,2}\cos 2\lambda + \bar{S}_{n,2}\sin 2\lambda) + \nu_{n,0}\bar{C}_{n,0}\Big]p_0, \tag{38}$$

$$T_{yzz} = -\sum_{n=2}^{N} \kappa_n(r)\sqrt{2n-1}\,(n+2)(n+3)\tau_{n,1} \times (\bar{S}_{n,1}\cos\lambda - \bar{C}_{n,1}\sin\lambda)p_1, \tag{39}$$

$$T_{zzz} = \sum_{n=2}^{N} \kappa_n(r)\sqrt{2n+1}\,(n+1)(n+2)(n+3)\,\bar{C}_{n,0}\,p_0. \tag{40}$$

Values of the factors p_0 and p_1 distinguish the North Pole from the South Pole. At the North Pole both p_0 and p_1 are equal to 1, at the South Pole their values are $p_0 = (-1)^n$ and $p_1 = (-1)^{n-1}$, respectively.

3. Derivation of the new non-singular expressions

In this section the new non-singular expressions of Eqs. (22)–(30) are derived. We start with the latitudinal derivatives of the associated Legendre functions of the first kind which are present in the series expansions as well as in the surface integrals used in the spherical harmonic analysis. Then, one by one, the derivation of the new non-singular expressions for the affected components of the third-order gravitational tensor is outlined.

3.1. Derivatives of the associated Legendre functions of the first kind

Firstly we formulate the expressions for the first-, second- and third-order latitudinal derivatives of the associated Legendre function. These expressions depend only on the associated

Legendre functions of the first kind themselves. The first-order derivative of the fully normalized associated Legendre function can be expressed as follows, see, e.g., Eshagh (2008, Eq. (18)):

$$\frac{d\bar{P}_{n,m}}{d\varphi} = -\frac{1}{2}\sqrt{(n+m)(n-m+1)\,\frac{1+\delta_{m-1,0}}{1+\delta_{m,0}}}\;\bar{P}_{n,m-1} + \frac{1}{2}\sqrt{(n-m)(n+m+1)\,\frac{1+\delta_{m+1,0}}{1+\delta_{m,0}}}\;\bar{P}_{n,m+1}, \quad m>0. \tag{41}$$

The symbol $\delta_{i,j}$ stands for the Kronecker delta. Note that this expression was originally derived in the non-normalized form by Kautzleben (1965) and Ilk (1983). For $m=0$ we get (Bosch, 2000):

$$\frac{d\bar{P}_{n,0}}{d\varphi} = \sqrt{\frac{1}{2}\,n\,(n+1)}\;\bar{P}_{n,1}. \tag{42}$$

Differentiating Eq. (41) twice, we get (Eshagh, 2008, Eq. (19)):

$$\frac{d^2\bar{P}_{n,m}}{d\varphi^2} = \frac{1}{4}\sqrt{(n+m)\,(n+m-1)\,(n-m+1)\,(n-m+2)\,\frac{1+\delta_{m-2,0}}{1+\delta_{m,0}}} \times \bar{P}_{n,m-2} - \frac{1}{4}[(n+m)\,(n-m+1)+(n+m+1)\,(n-m)]\bar{P}_{n,m} + \frac{1}{4}\sqrt{(n+m+2)\,(n+m+1)\,(n-m)\,(n-m-1)\,\frac{1+\delta_{m+2,0}}{1+\delta_{m,0}}} \times \bar{P}_{n,m+2}, \quad m>1. \tag{43}$$

For $m=0$ we can use the generating equation for the associated Legendre function (Heiskanen and Moritz, 1967, Eqs. (1)–(60)):

$$P_{n,m}(t) = (1-t^2)^{m/2}\frac{d^m}{dt^m}P_{n,0}(t), \tag{44}$$

where the substitution $t=\sin\varphi$ is used. Note that in Eq. (44) the non-normalized associated Legendre functions are used. After some simplification and normalization we obtain:

$$\frac{d^2\bar{P}_{n,0}}{d\varphi^2} = \frac{1}{4}\sqrt{2\,(n+2)\,(n+1)\,n\,(n-1)}\;\bar{P}_{n,2} - \frac{1}{2}n\,(n+1)\,\bar{P}_{n,0}. \tag{45}$$

The expression for $m=1$ can be obtained by substituting Eq. (42) into Eq. (41). The result of this substitution still contains the derivative of the associated Legendre function of the first kind which can be eliminated by applying Eq. (41):

$$\frac{d^2\bar{P}_{n,1}}{d\varphi^2} = \frac{1}{4}\sqrt{(n+3)\,(n+2)\,(n-1)\,(n-2)}\;\bar{P}_{n,3} - \left[\frac{1}{2}n\,(n+1)+\frac{1}{4}(n+2)\,(n-1)\right]\bar{P}_{n,1}. \tag{46}$$

The expression for the third-order derivative can be derived by exploiting Eq. (41) in Eq. (43). Thus, we obtain:

$$\frac{d^3\bar{P}_{n,m}}{d\varphi^3} = -\frac{1}{8}\sqrt{(n+m)(n+m-1)(n+m-2)(n-m+1)(n-m+2)(n-m+3)\,\frac{1+\delta_{m-3,0}}{1+\delta_{m,0}}}\;\bar{P}_{n,m-3} + \frac{1}{8}\sqrt{(n+m)(n-m+1)\frac{1+\delta_{m-1,0}}{1+\delta_{m,0}}}\,[(n+m-1)(n-m+2)+(n+m)(n-m+1)+(n+m+1)(n-m)]\,\bar{P}_{n,m-1} - \frac{1}{8}\sqrt{(n-m)(n+m+1)\frac{1+\delta_{m+1,0}}{1+\delta_{m,0}}}\,[(n+m)(n-m+1)+(n+m+1)(n-m)+(n+m+2)(n-m-1)]\,\bar{P}_{n,m+1} + \frac{1}{8}\sqrt{(n+m+3)(n+m+2)(n+m+1)(n-m)(n-m-1)(n-m-2)\frac{1+\delta_{m+3,0}}{1+\delta_{m,0}}}\;\bar{P}_{n,m+3}, \quad m>2. \tag{47}$$

The corresponding expressions for $m\le 2$ are derived by analogous techniques as in the previous two cases. The resulting relations are:

$$\frac{d^3\bar{P}_{n,0}}{d\varphi^3} = -\left[\frac{3}{4}(n+2)(n-1)+1\right]\sqrt{\frac{1}{2}n(n+1)}\,\bar{P}_{n,1} + \frac{1}{4}\sqrt{\frac{1}{2}(n+3)(n+2)(n+1)n(n-1)(n-2)}\;\bar{P}_{n,3}, \tag{48}$$

$$\frac{d^3\bar{P}_{n,1}}{d\varphi^3} = \frac{1}{8}\sqrt{2n(n+1)}\left[2n(n+1)+(n+2)(n-1)\right]\bar{P}_{n,0} - \frac{1}{8}\sqrt{(n+2)(n-1)}\left[2n(n+1)+(n+2)(n-1)+(n+3)(n-2)\right]\bar{P}_{n,2} + \frac{1}{8}\sqrt{(n+4)(n+3)(n+2)(n-1)(n-2)(n-3)}\;\bar{P}_{n,4}, \tag{49}$$

$$\frac{d^3\bar{P}_{n,2}}{d\varphi^3} = \frac{1}{8}\sqrt{(n+2)(n-1)}\left[2n(n+1)+(n+2)(n-1)+(n+3)(n-2)\right]\bar{P}_{n,1} - \frac{1}{8}\sqrt{(n+3)(n-2)}\left[(n+2)(n-1)+(n+3)(n-2)+(n+4)(n-3)\right]\bar{P}_{n,3} + \frac{1}{8}\sqrt{(n+5)(n+4)(n+3)(n-2)(n-3)(n-4)}\;\bar{P}_{n,5}. \tag{50}$$

3.2. Deriving the non-singular expressions

Now we will eliminate the terms which are singular at the poles, i.e., the terms which contain cosine functions of the spherical latitude φ in the denominator. According to Eshagh (2008, Eq. (26)) we can write:

$$\frac{1}{\cos\varphi}\,\bar{P}_{n,m} = \frac{1}{2m}\sqrt{\frac{(2n+1)(1+\delta_{m-1,0})}{(2n-1)(1+\delta_{m,0})}\,(n+m)\,(n+m-1)}\;\bar{P}_{n-1,m-1} + \frac{1}{2m}\sqrt{\frac{(2n+1)(1+\delta_{m+1,0})}{(2n-1)(1+\delta_{m,0})}\,(n-m)\,(n-m-1)}\,\bar{P}_{n-1,m+1}, \quad m>0. \tag{51}$$

For the inverse second power of $\cos\varphi$ we have (Eshagh, 2008, Eq. (27)):

$$\frac{1}{\cos^2\varphi}\bar{P}_{n,m} = \frac{1}{4m(m-1)}\sqrt{\frac{1+\delta_{m-2,0}}{1+\delta_{m,0}}(n-m+1)(n-m+2)(n+m)(n+m-1)}\;\bar{P}_{n,m-2} + \frac{1}{4m}\left[\frac{(n+m)(n+m-1)}{m-1}+\frac{(n-m)(n-m-1)}{m+1}\right]\bar{P}_{n,m} + \frac{1}{4m(m+1)}\sqrt{\frac{1+\delta_{m+2,0}}{1+\delta_{m,0}}(n+m+1)(n+m+2)(n-m)(n-m-1)}\;\bar{P}_{n,m+2}, \quad m>1. \tag{52}$$

By substituting Eq. (51) into Eq. (52) we obtain the expression for the inverse of the third power of cos φ:

$$\frac{1}{\cos^3\varphi}\bar{P}_{n,m} = \frac{1}{8m(m-1)(m-2)}\sqrt{\frac{(2n+1)(1+\delta_{m-3,0})}{(2n-1)(1+\delta_{m,0})}(n-m+1)(n-m+2)(n+m)(n+m-1)(n+m-2)(n+m-3)}\;\bar{P}_{n-1,m-3} + \frac{1}{8m}\sqrt{\frac{2n+1}{2n-1}(n+m)(n+m-1)}\left[\frac{3m^2+m(2n+3)+3n^2+n-6}{(m-1)(m+2)}\right]\bar{P}_{n-1,m-1} + \frac{1}{8m}\sqrt{\frac{2n+1}{2n-1}(n-m)(n-m-1)}\left[\frac{3m^2-m(2n+3)+3n^2+n-6}{(m+1)(m-2)}\right]\bar{P}_{n-1,m+1} + \frac{1}{8m(m+1)(m+2)}\sqrt{\frac{(2n+1)(1+\delta_{m+3,0})}{(2n-1)(1+\delta_{m,0})}(n+m+1)(n+m+2)(n-m)(n-m-1)(n-m-2)(n-m-3)} \times \bar{P}_{n-1,m+3}, \quad m>2. \tag{53}$$

The corresponding expressions for $m = 0, 1, 2$ are solved separately for each of the components containing the singular terms. They are often taken from Petrovskaya and Vershkov (2006). Alternatively the expressions are developed directly from the combination of factors which appear in the singular (original) expressions. The cases when the sine and tangent functions appear in the equations are solved separately for each component.

- *Derivation of the expression for the component T_{xxx}*: After substituting Eqs. (41) and (47) into Eq. (12) and after some simplifications we obtain Eq. (22).
- *Derivation of the expressions for the components T_{xxz} and T_{yyz}*: According to Petrovskaya and Vershkov (2006, Eq. (10)) we can write:

$$(n+1)\bar{P}_{n,m} - \frac{d^2\bar{P}_{n,m}}{d\varphi^2} = k_{n,m}\bar{P}_{n,m-2} + l_{n,m}\bar{P}_{n,m} + o_{n,m}\bar{P}_{n,m+2}, \tag{54}$$

and after substituting this identity into Eq. (14) we obtain Eq. (24). The non-singular expression for T_{yyz} is derived using the derivative of the Laplace equation with respect to z:

$$T_{xxz} + T_{yyz} + T_{zzz} = 0. \tag{55}$$

After substituting Eqs. (24) and (21) for the components T_{xxz} and T_{zzz} into Eq. (55), we get Eq. (29) for the component T_{yyz}.

- *Derivation of the expressions for the components T_{xyy} and T_{xzz}*: The derivation of the new non-singular formula for T_{xzz}, see Eq. (27), can simply be performed by substituting Eq. (41) into Eq. (17). The expression for T_{xyy} can be derived using the derivative of the Laplace equation with respect to x:

$$T_{xxx} + T_{xyy} + T_{xzz} = 0. \tag{56}$$

Substituting Eqs. (22) and (27) for the components T_{xxx} and T_{xzz} into Eq. (56), we obtain Eq. (25) for the component T_{xyy}.

- *Derivation of the expression for the component T_{xyz}*: According to Petrovskaya and Vershkov (2006, Eq. (12)) we can write:

$$m\left(\frac{\sin\varphi}{\cos^2\varphi}\bar{P}_{n,m} + \frac{1}{\cos\varphi}\frac{d\bar{P}_{n,m}}{d\varphi}\right) = s_{n,m}\bar{P}_{n-1,m-2} + v_{n,m}\bar{P}_{n-1,m} + w_{n,m}\bar{P}_{n-1,m+2}. \tag{57}$$

Substituting Eq. (57) into Eq. (16) we obtain Eq. (26).

- *Derivation of the expression for the component T_{yyy}*: According to Eshagh (2010, Eq. (51)) we can write:

$$\tan\varphi\frac{d\bar{P}_{n,m}}{d\varphi} = -\frac{1}{4(m-1)}\sqrt{\frac{1+\delta_{m-2,0}}{1+\delta_{m,0}}(n+m)(n-m+1)(n-m+2)(n+m-1)}\;\bar{P}_{n,m-2} + \left[\frac{(n+m+1)(n-m)}{4(m+1)} - \frac{(n+m)(n-m+1)}{4(m-1)}\right]\bar{P}_{n,m} + \frac{1}{4(m+1)}\sqrt{\frac{1+\delta_{m+2,0}}{1+\delta_{m,0}}(n-m)(n-m-1)(n+m+2)(n+m+1)} \times \bar{P}_{n,m+2} \quad m>1. \tag{58}$$

By substituting Eqs. (58), (51) and (53) into Eq. (18) we obtain Eq. (28) for $m \neq 1, 2$. The expressions for $m = 1, 2$ can be deduced using the relation (Ilk, 1983):

$$\sin\varphi\cos\varphi\frac{dP_{n,m}}{d\varphi} = -m\,P_{n,m} - (n+1)\cos^2\varphi P_{n,m} + \cos\varphi P_{n+1,m+1}. \tag{59}$$

Note that in this relation the non-normalized associated Legendre functions are used. The part of Eq. (18) which depends on latitude φ can be rewritten as:

Table 1
Statistics of the disturbing gravitational curvatures in the polar regions $80 < \varphi < 90$ arc-deg and $-90 < \varphi < -80$ arc-deg. All grids were synthesised from EGM08 to the maximum degree n=360 over the 10 arc-min grid at the mean sphere of radius a=6378136.3 m. All values are given in 10^{-13} m^{-1} s^{-2}.

Component	North Pole				South Pole			
	mean	std	min	max	mean	std	min	max
T_{xxy}	0.00	0.27	−2.03	1.45	0.00	0.12	−0.49	0.56
T_{xyy}	0.00	0.27	−2.05	1.58	0.00	0.12	−0.61	0.51
T_{xyz}	0.00	0.38	−2.62	2.40	0.00	0.17	−0.71	0.94
T_{yyy}	0.00	0.64	−3.93	3.26	0.00	0.29	−1.46	1.24
T_{yyz}	0.00	0.67	−4.00	4.09	0.00	0.32	−1.44	1.66
T_{yzz}	0.00	0.80	−3.84	5.62	0.00	0.36	−1.53	1.95

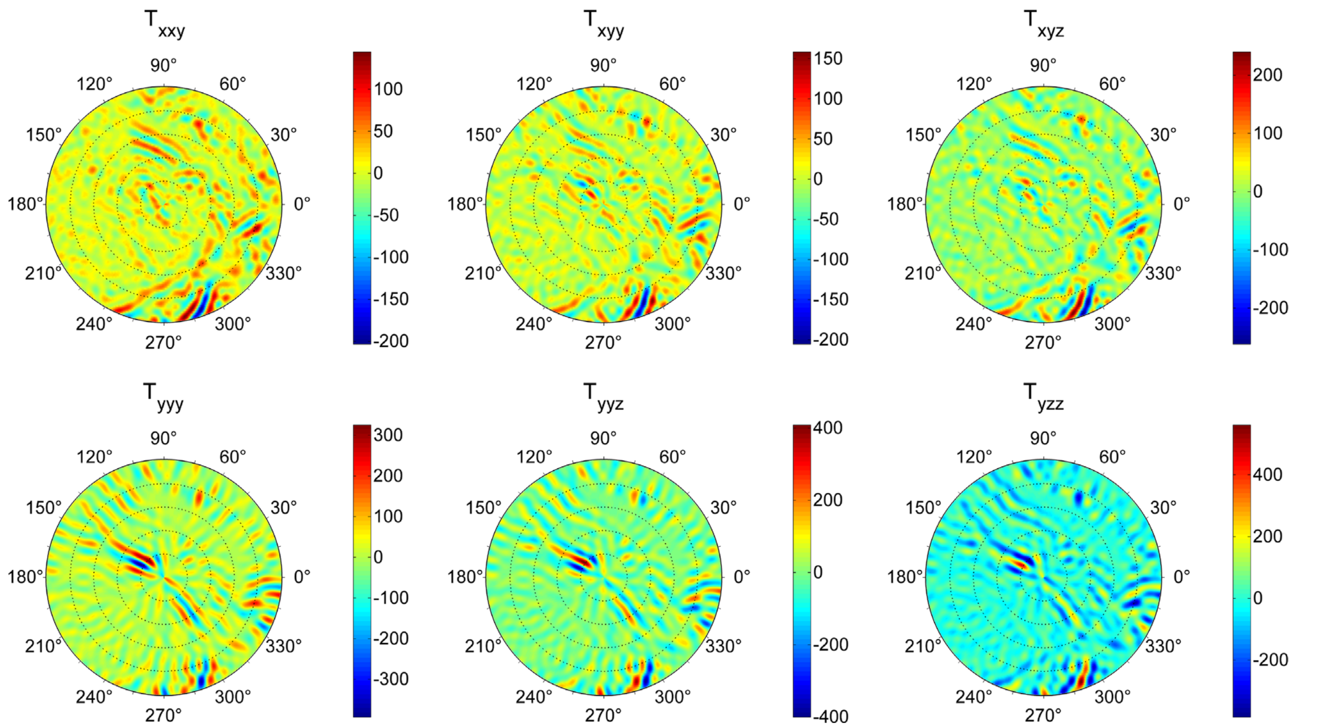

Fig. 1. Values of the disturbing gravitational curvatures over the North Pole region. Units are 10^{-15} m^{-1} s^{-2}.

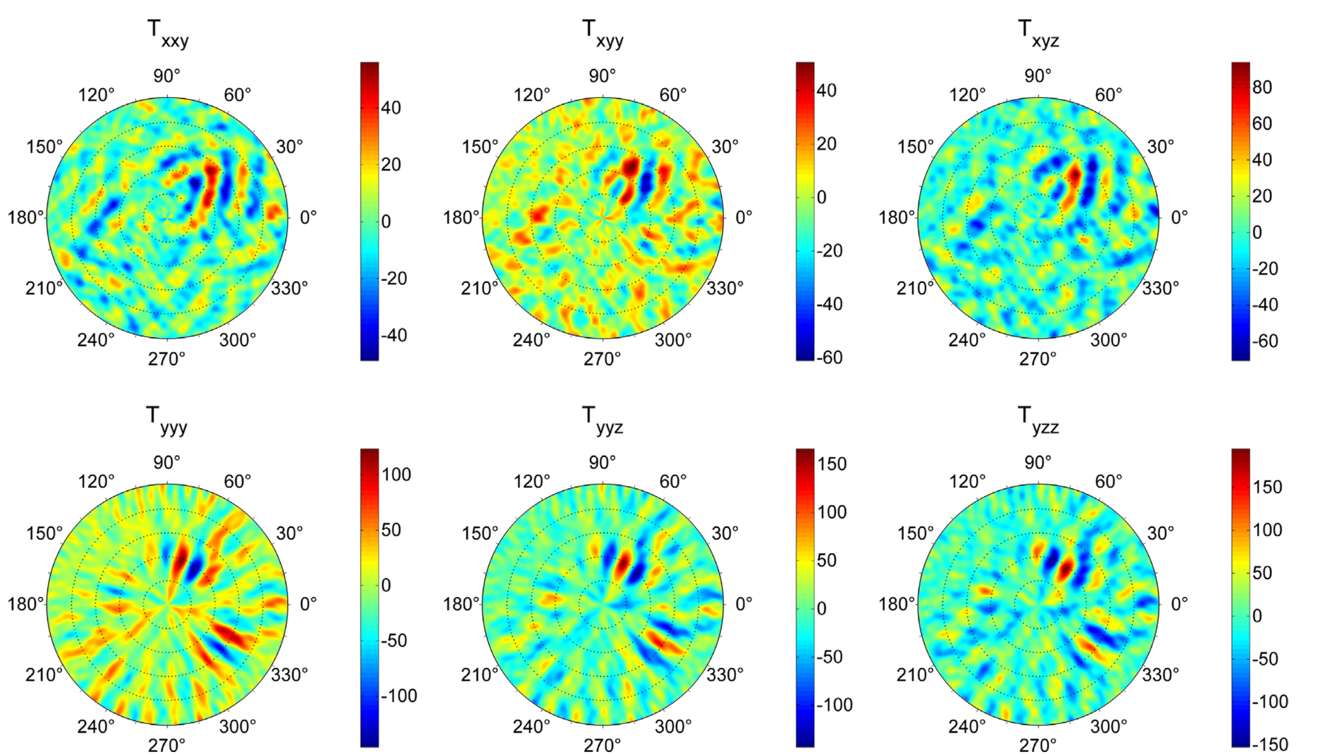

Fig. 2. Values of the disturbing gravitational curvatures over the South Pole region. Units are 10^{-15} m^{-1} s^{-2}.

$$\frac{1}{\cos^3\varphi}\left[(m^2+2)P_{n,m} + 3(n+1)\cos^2\varphi P_{n,m} + 3\sin\varphi\cos\varphi\frac{\mathrm{d}P_{n,m}}{\mathrm{d}\varphi}\right]. \tag{60}$$

By substituting Eq. (59) into Eq. (60) we obtain:

$$\frac{1}{\cos^3\varphi}\left[(m-2)(m-1)P_{n,m} + 3\cos\varphi\frac{\mathrm{d}P_{n,m}}{\mathrm{d}\varphi}\right] = \frac{3}{\cos^2\varphi}P_{n+1,m+1} \quad \text{for } m=1 \text{ or } m=2. \tag{61}$$

The singular term $\cos^{-2}\varphi$ in Eq. (61) may be eliminated by using Eq. (52); after normalization we get Eq. (28) for $m = 1, 2$.

Table 2
Statistics of the differences between values of the disturbing gravitational curvatures computed by the singular expressions of Eqs. (12)–(20) and by the new non-singular expressions of Eqs. (22)–(30) over the non-polar region $-80 < \varphi < 80$ arc-deg. The grids were synthesized from EGM08 to the maximum degree $n=360$ over the 10 arc-min grid at the mean sphere of radius $a=6378136.3$ m. All values are given in 10^{-25} m^{-1} s^{-2}.

Component	mean	std	min	max
T_{xxx}	0.00	2.24	−4.16	4.76
T_{xxy}	0.00	0.03	−2.13	1.66
T_{xxz}	0.04	2.25	−6.14	4.41
T_{xyy}	0.00	0.75	−2.49	2.07
T_{xyz}	0.00	0.05	−2.97	3.05
T_{xzz}	0.00	2.99	−6.00	5.42
T_{yyy}	0.00	0.11	−4.90	5.75
T_{yyz}	1.19	1.63	−6.04	4.66
T_{yzz}	0.00	1.29	−6.90	6.25

Table 3
Statistics of the Laplacians of the disturbing gravitational curvatures computed by the new non-singular expressions over the polar regions. All grids were synthesized from EGM08 to the maximum degree $n=360$ over the 10 arc-min grid at the mean sphere of radius $a=6378136.3$ m. All values are given in 10^{-29} m^{-1} s^{-2}.

Laplacian	North Pole				South Pole			
	mean	std	min	max	mean	std	min	max
$T_{xxx}+T_{xyy}+T_{xzz}$	0.02	5.76	−35.97	55.54	0.21	3.94	−20.19	24.61
$T_{xxy}+T_{yyy}+T_{yzz}$	0.00	5.86	−32.82	40.23	0.00	4.17	−19.56	22.09
$T_{xxz}+T_{yyz}+T_{zzz}$	0.20	7.63	−60.58	50.49	0.03	5.26	−30.29	31.55

Table 4
Global statistics of the disturbing gravitational curvatures. All grids were synthesized from EGM08 to the maximum degree $n=360$ over the 10 arc-min grid at the mean sphere of radius $a=6378136.3$ m. All values are given in 10^{-13} m^{-1} s^{-2}.

Component	mean	std	min	max
T_{xxx}	0.00	1.19	−24.08	25.96
T_{xxy}	0.00	0.50	−10.72	12.46
T_{xxz}	0.00	1.29	−24.18	31.74
T_{xyy}	0.00	0.49	−11.73	12.15
T_{xyz}	0.00	0.70	−15.49	17.37
T_{xzz}	0.00	1.47	−36.16	31.09
T_{yyy}	0.00	1.14	−25.84	28.40
T_{yyz}	0.00	1.24	−29.68	29.73
T_{yzz}	0.00	1.42	−34.80	30.13
T_{zzz}	0.00	2.05	−57.23	40.27

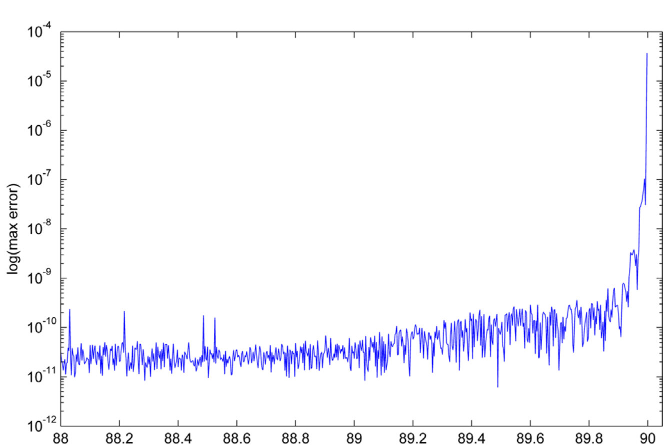

Fig. 3. Differences for T_{yyy} component between original and non-singular expressions close to the North Pole. Units are 10^{-15} m^{-1} s^{-2}.

- *Derivation of the expressions for the components T_{xxy} and T_{yzz}*: The non-singular expression for the T_{yzz} component, see Eq. (30), is derived by substituting Eq. (51) into Eq. (20). The expression for T_{xxy} is derived using the derivative of the Laplace equation with respect to y:

$$T_{xxy} + T_{yyy} + T_{yzz} = 0. \tag{62}$$

Substituting Eqs. (28) and (30) for the components T_{yyy} and T_{yzz} into Eq. (62), we obtain Eq. (23) for the component T_{xxy}.

4. Numerical investigations

The new non-singular formulas were implemented with computer-based programs and then verified. In all numerical experiments presented herein the Earth Gravitational Model 2008 (EGM08, Pavlis et al., 2012) was used. All presented values were synthesized over the equiangular 10 arc-min grids from the geopotential coefficients up to the degree and order 360 at the mean sphere of radius $a=6378136.3$ m.

All programs were written in Matlab. Computations were optimized through application of the lumped-coefficient method (Colombo, 1981). The associated Legendre functions of the first kind were computed using the standard forward column method (Holmes and Featherstone, 2002) which is sufficiently accurate for computations up to the degree 1800.

In Table 1 statistics of the disturbing gravitational curvatures computed over the two polar regions, i.e., the North Pole region of $80 < \varphi < 90$ arc-deg and the South Pole region of $-90 < \varphi < -80$ arc-deg, can be found. Only the gravitational curvatures with singularities in the original formulas are presented in the table. Figs. 1 and 2 then show numerical values of the respective disturbing gravitational curvatures based on the new non-singular expressions. In total, the disturbing gravitational curvatures were synthesized at 61×2160 grid points over each of the two polar regions. The magnitudes of the disturbing gravitational curvatures over both polar regions in terms of their standard deviations reached the level of 10^{-14} m^{-1} s^{-2}.

In the next experiment for the nine components, the disturbing gravitational curvatures computed by the new non-singular expressions of Eqs. (22)–(30) over the non-polar area defined by the latitude band of $-80 < \varphi < 80$ arc-deg were compared with values of the respective disturbing gravitational curvatures computed by the original singular expressions of Eqs. (12)–(20). Values of the disturbing gravitational curvatures from both (singular and non-singular) sets of equations were synthesized at 961×2160 grid points. Values of their differences can be found in Table 2. As it can be seen, the values of the differences are everywhere smaller than 10^{-24} m^{-1} s^{-2}. These tests validated the correctness of the derived non-singular equations and their computer realization (computer codes).

In the last numerical experiment, all three Laplace conditions, see Eqs. (55), (56) and (62), valid for the diagonal gravitational curvatures, were checked at the two polar regions. Statistics of the three Laplacians can be found in Table 3. The values of the Laplacians are everywhere smaller than 10^{-27} m^{-1} s^{-2}. All these numerical tests successfully validated applicability and correctness of the new non-singular formulas (values at the poles, values of the differences) as well as of their computer realizations (values of the Laplacians).

Global statistics for all ten gravitational curvatures were also computed and can be found in Table 4. The signal powers of the individual components can be estimated from their standard deviations. As expected, the largest signal level is for the T_{zzz} component followed by T_{xzz}, T_{yzz}, T_{xxz}, T_{yyz}, T_{xxx}, T_{yyy}, T_{xyz}, T_{xxy} and T_{xyy}. As it can also be noticed, there are six pairs of the components with nearly the same signal power – T_{xzz} and T_{yzz}, T_{xxz} and T_{yyz}, T_{xxx} and T_{yyy}, and T_{xxy} and T_{xyy}.

At Fig. 3 differences between original and new non-singular formulas around the North Pole can be seen. For each latitude the maximum difference on each parallel is depicted. Both grids were computed with the 10 arc-sec step in latitude and 1 arc-deg step in longitude. Note that the difference starts to grow for latitudes larger than 89 arc-deg and increases rapidly over the last 10 arc-min around the North Pole (North Pole itself is left out from this graph). Depicted differences are those of the T_{yyy} component, but a similar behavior can be observed for all other components with singularities in the original form.

5. Conclusions

The new expressions for the spherical harmonic synthesis of the gravitational curvatures from available global gravitational models in the local north-oriented reference frame were derived in this paper. In contrast to the conventional expansions of Eqs. (12)–(20), the new expansions presented by Eqs. (22)–(30) have the following advantages:

- To synthesize the third-order gradients of the gravitational potential (gravitational curvatures) recurrent relations between the associated Legendre functions of the first kind only are required, while in the case of the conventional expressions recurrent relations are based on the first-, second- and third-order latitudinal derivatives of these functions. Consequently, the computer algorithms based on the new expressions are simpler which also results in higher speed and stability of numerical evaluations which would be convenient for processing large amounts of observational data yielded possibly by future satellite sensors.
- The new expressions are free of singularities that are present in the conventional formulas since respective numerical coefficients of the associated Legendre functions and their derivatives contain singular terms dependent on the latitude. This property results in the worldwide applicability of the new expressions including polar regions where the conventional formulas run into numerical problems when approaching the poles.

The coefficients of the associated Legendre functions of the first kind and their latitudinal derivatives in Eqs. (12)–(20) contain singular terms dependent on the latitude. Thus, these expressions cannot be used in polar areas. In the new expressions these transformations are included and the numerical coefficients of the respective spherical harmonic series depend only on the degree and order of spherical harmonics. The new expressions have been implemented in the computer program and we also provide the program for potential users. The third-order latitudinal derivatives of the associated Legendre function would also be important for the spherical harmonic analysis of the gravitational curvature fields once they become observable.

Acknowledgments

E. Hamáčková was supported by the project SGS-2013-024. M. Šprlák and P. Novák were supported by the project GA15-08045S of the Czech Science Foundation. M. Pitoňák was supported by the motivation system POSTDOC 2015 of the University of West Bohemia. Thoughtful and constructive comments of Prof. Mehdi Eshagh and one anonymous reviewer are gratefully acknowledged.

Appendix A

The numerical constants, that appear in Eqs. (22)–(30), have the following forms:

$$a_{n,m} = \frac{1}{8}\sqrt{(1+\delta_{m-3,0})(n+m)(n+m-1)(n+m-2)}\sqrt{(n-m+1)(n-m+2)(n-m+3)},\ m>2,\quad a_{n,m}=0,\quad m=0,1,2. \tag{63}$$

$$\begin{aligned} b_{n,m} &= \frac{3}{8}\sqrt{(n+m)(n-m+1)}\,(m+n+2)(m-n-3), \quad m>2,\\ b_{n,m} &= 0, \quad m=0,\\ b_{n,m} &= -\frac{3}{8}\sqrt{2n(n+1)}\,(n+2)(n+3), \quad m=1,\\ b_{n,m} &= -\frac{1}{2}\sqrt{(n-1)(n+2)}\,(n+1)(n+3), \quad m=2, \end{aligned} \tag{64}$$

$$\begin{aligned} c_{n,m} &= -\frac{3}{8}\sqrt{(n-m)(n+m+1)}\,(m-n-2)(m+n+3), \quad m>2,\\ c_{n,m} &= \frac{3}{4}\sqrt{\frac{1}{2}n(n+1)}\,(n+2)(n+3), \quad m=0,\\ c_{n,m} &= \frac{1}{2}\sqrt{(n-1)(n+2)}\,(n+1)(n+3), \quad m=1,\\ c_{n,m} &= \frac{3}{8}\sqrt{(n-2)(n+3)}\,n(n+5),\ m=2, \end{aligned} \tag{65}$$

$$e_{n,m} = -\frac{1}{8m}\sqrt{\frac{2n+1}{2n-1}(1+\delta_{m-3,0})(m+n)(m+n-1)(m+n-2)(m+n-3)(n-m+1)(n-m+2)},\ m>2,\ e_{n,m}=0, \quad m=1,2\,, \tag{67}$$

$$\begin{aligned} d_{n,m} &= -\frac{1}{8}\sqrt{(n-m)(n-m-1)(n-m-2)(n+m+1)(n+m+2)(n+m+3)},\\ &\quad m>0,\\ d_{n,m} &= -\frac{1}{4}\sqrt{\frac{1}{2}n(n-1)(n-2)(n+1)(n+2)(n+3)}, \quad m=0, \end{aligned} \tag{66}$$

$$f_{n,m} = -\frac{1}{8m}\sqrt{\frac{2n+1}{2n-1}(m+n)(m+n-1)}\,[3m^2 - m(2n+3) - n^2 - 3(n+2)], \quad m>2,$$

$$f_{n,m} = \frac{1}{8}\sqrt{2n(n+1)\,\frac{2n+1}{2n-1}}\,(n+2)(n+3), \quad m=1,$$

$$f_{n,m} = \frac{1}{8}\sqrt{(n+1)(n+2)\,\frac{2n+1}{2n-1}}\,n(n+3), \quad m=2, \tag{68}$$

$$g_{n,m} = -\frac{1}{8m}\sqrt{\frac{2n+1}{2n-1}\,(m-n)(m-n+1)}\,[3m^2+m(2n+3)-n^2 - 3(n+2)], \quad m>2,$$

$$g_{n,m} = 0, \quad m=1,$$

$$g_{n,m} = \frac{1}{16}\sqrt{(n-2)(n-3)\,\frac{2n+1}{2n-1}}\,(n+3)(n-4), \; m=2, \tag{69}$$

$$r_{n,m} = -\frac{1}{8}\sqrt{(n-2)(n+3)}\,(n^2+5n+24), \quad m=2, \tag{75}$$

$$s_{n,m} = \frac{1}{4}\sqrt{\frac{2n+1}{2n-1}(1+\delta_{m-2,0})(n+m)(n+m-1)(n+m-2)(n-m+1)}, \quad m>1, \; s_{n,m}=0, \quad m=1, \tag{76}$$

$$v_{n,m} = -\frac{m}{2}\sqrt{\frac{2n+1}{2n-1}\,(n+m)(n-m)}, \quad m>1,$$

$$v_{n,m} = \frac{1}{4}\sqrt{\frac{2n+1}{2n-1}\,(n+1)(n-1)}(n+2), \quad m=1, \tag{77}$$

$$w_{n,m} = -\frac{1}{4}\sqrt{\frac{2n+1}{2n-1}(n-m)(n-m-1)(n-m-2)(n+m+1)}, \tag{78}$$

$$h_{n,m} = -\frac{1}{8m}\sqrt{\frac{2n+1}{2n-1}\,(n-m)(n+m+1)(n+m+2)(n-m-1)(n-m-2)(n-m-3)}\; m>0, \tag{70}$$

$$k_{n,m} = -\frac{1}{4}\sqrt{(1+\delta_{m-2,0})(m+n)(n+m-1)(n-m+1)(n-m+2)}, \quad m>1, \; k_{n,m}=0, \quad m=0,1, \tag{71}$$

$$l_{n,m} = \frac{1}{2}\,[(n+1)(n+2)-m^2], \quad m>1,$$

$$l_{n,m} = \frac{1}{4}\,(n+2)(3n+1), \quad m=1, \tag{72}$$

$$o_{n,m} = -\frac{1}{4}\sqrt{(n-m)(n-m-1)(n+m+1)(n+m+2)}, \quad m>0,$$

$$o_{n,m} = -\frac{1}{4}\sqrt{2(n-1)n(n+1)(n+2)}, \quad m=0, \tag{73}$$

$$q_{n,m} = \frac{1}{8}\sqrt{(1+\delta_{m-1,0})(n+m)(n-m+1)}\,[3m(m-1) + (n+2)(n+3)], \quad m\neq 0,2,$$

$$q_{n,m} = 0, \quad m=0,$$

$$q_{n,m} = \frac{1}{2}\sqrt{(n-1)(n+2)}\,(n+3), \quad m=2, \tag{74}$$

$$r_{n,m} = -\frac{1}{8}\sqrt{(n-m)(n+m+1)}\,[3m(m+1)+(n+2)(n+3)], \quad m>2,$$

$$r_{n,m} = -\frac{1}{4}\sqrt{\frac{1}{2}n(n+1)}\,(n+2)(n+3), \quad m=0,$$

$$r_{n,m} = -\frac{1}{2}\sqrt{(n-1)(n+2)}\,(n+3), \quad m=1,$$

$$\beta_{n,m} = -\frac{1}{2}\sqrt{(1+\delta_{m-1,0})(n+m)(n-m+1)}, \quad m>0,$$

$$\beta_{n,m} = 0, \quad m=0, \tag{79}$$

$$\gamma_{n,m} = \frac{1}{2}\sqrt{(n-m)(n+m+1)}, \quad m>0,$$

$$\gamma_{n,m} = \sqrt{\frac{1}{2}\,n(n+1)}, \quad m=0, \tag{80}$$

$$\epsilon_{n,m} = \frac{1}{8m}\sqrt{\frac{2n+1}{2n-1}(m+n)(m+n-1)}\,[3m^2-m(2n+3)+3n^2 + 17n+18], \quad m>2,$$

$$\epsilon_{n,m} = \frac{3}{4m(m+1)}\sqrt{\frac{2n+1}{2n-1}(1+\delta_{m-1,0})(m+n)(m+n-1)} \times (n+m+1)(n+m+2), \quad m=1,2, \tag{81}$$

$$\mu_{n,m} = \frac{1}{8m}\sqrt{\frac{2n+1}{2n-1}(m-n)(m-n+1)}\,[3m^2+m(2n+3)+3n^2 + 17n+18], \quad m>2,$$

$$\mu_{n,m} = \frac{3}{2m(m+2)}\sqrt{\frac{2n+1}{2n-1}(n-m)(n-m-1)} \times (n+m+1)(n+m+2), \quad m=1,2, \tag{82}$$

$$\nu_{n,m} = -\frac{1}{2}\,[(n+1)(n+2)+m^2], \quad m>1,$$

$$\nu_{n,m} = -\frac{1}{4}\,(n+2)(n+3), \quad m=1, \tag{83}$$

$$\tau_{n,m} = \frac{1}{2m}\sqrt{\frac{2n+1}{2n-1}(1+\delta_{m-1,0})(n+m)(n+m-1)}, \quad m > 0, \tag{84}$$

$$\chi_{n,m} = \frac{1}{2m}\sqrt{\frac{2n+1}{2n-1}(n-m)(n-m-1)} \quad m > 0. \tag{85}$$

Appendix B. Supplementary data

Supplementary data associated with this paper can be found in the online version at http://dx.doi.org/10.1016/j.cageo.2015.12.011.

References

Abdelrahman, E.-S.M., El-Araby, H.M., El-Araby, T.M., Abo-Ezz, E.R., 2003. A least-squares derivatives analysis of gravity anomalies due to faulted thin slabs. Geophysics 68, 535–543.

Albertella, A., Migliaccio, F., Sansó, F., Tscherning, C.C., 2000. The space-wise approach—overall scientific data strategy. In: Sünkel, H. (Ed.), From Eötvös to Milligal. Final Report of the ESA/ESTEC Contract No. 13329/98/NL/GD, pp. 267–298.

Ardalan, A.A., Grafarend, E.W., 2001. Ellipsoidal geoidal undulations (ellipsoidal Bruns formula). J. Geodesy 75, 544–552.

Balakin, A.B., Daishev, R.A., Murzakhanov, Z.G., Skochilov, A.F., 1997. Laser-interferometric detector of the first, second and third derivatives of the potential of the Earth gravitational field. Izv. vyss. uchebnykh zaved. ser. Geol. Razved. 1, 101–107, in Russian.

Beiki, M., 2010. Analytic signals of gravity gradient tensor and their application to estimate source location. Geophysics 75, 159–174.

Bosch, W., 2000. On the computation of derivatives of Legendre functions. Phys. Chem. Earth 25, 655–659.

Brieden. P., Müller, J., Flury, J., Heinzel, G., 2010. The Mission OPTIMA—Novelties and Benefit. Geotechnologien, Science Report, No. 17, Potsdam, Germany, pp. 134–139.

Casotto, S., Fantino, E., 2009. Gravitational gradients by tensor analysis with application to spherical coordinates. J. Geodesy 83, 621–634.

Colombo, O.L., 1981. Numerical Methods for Harmonic Analysis on the Sphere. Report No. 310. Department of Geodetic Science and Surveying, The Ohio State University, Columbus, OH, USA, 140 pp.

Cunningham, L., 1970. On the computation of the spherical harmonic terms needed during the numerical integration of the orbital motion of an artificial satellite. Celest. Mech. 2, 207–216.

Du, J., Chen, C., Lesur, V., Wang, L., 2015. Non-singular spherical harmonic expressions of geomagnetic vector and gradient tensor fields in the local north-oriented reference frame. Geosci. Model Dev. 8, 1979–1990.

Eppelbaum, L.V., 2011. Review of environmental and geological microgravity applications and feasibility of its employment at archaeological sites in Israel. Int. J. Geophys. 2011, 9 pp.

ESA, 1999. Gravity Field and Steady-State Ocean Circulation Mission. ESA SP-1233 (1), Report for Mission Selection of the Four Candidate Earth Explorer Missions. ESA Publications Division, 217 pp.

Eshagh, M., 2008. Non-singular expressions for the vector and the gradient tensor of gravitation in a geocentric spherical frame. Comput. Geosci. 34, 1762–1768.

Eshagh, M., 2010. Alternative expressions for gravity gradients in local north-oriented frame and tensor spherical harmonics. Acta Geophys. 58, 215–243.

Fantino, E., Casotto, S., 2009. Methods of harmonic synthesis for global geopotential models and their first-, second- and third-order gradients. J. Geodesy 83, 595–619.

Fedi, M., Florio, G., 2001. Detection of potential field source boundaries by enhanced horizontal derivative method. Geophys. Prospect. 49, 40–58.

Fukushima, T., 2012. Numerical computation of spherical harmonics of arbitrary degree and order by extending exponent of floating point numbers: II first-, second-, and third-order derivatives. J. Geodesy 86, 1019–1028.

Fukushima, T., 2013. Recursive computation of oblate spheroidal harmonics of the second kind and their first-, second-, and third-order derivatives. J. Geodesy 87, 303–309.

Grafarend, E.W., 1997. Field lines of gravity, their curvature and torsion, the Lagrangian and the Hamilton equations of the plumbline. Ann. Geofis. 40, 1233–1247.

Hafez, M., El-Qady, G., Awad, S., Elsayed, E.-S., 2006. Higher derivative analysis for the interpretation of self-potential profiles at southern part of Nile delta, Egypt. Arab. J. Sci. Eng. 31, 129–145.

Heiskanen, W.A., Moritz, H., 1967. Physical Geodesy. Freeman and Co., San Francisco, USA.

Holmes, S.A., Featherstone, W.E., 2002. A unified approach to the Clenshaw summation and the recursive computation of very high degree and order normalised associated Legendre functions. J. Geodesy 76, 279–299.

Ilk, K.H., 1983. Ein Beitrag zur Dynamik ausgedehnter Körper – Gravitationswechselwirkung. Deutsche Geodätische Kommission, Reihe C, Heft Nr. 288, München, Germany (in German).

Jacoby, W., Smilde, P.L., 2009. Gravity Interpretation: Fundamentals and Application of Gravity Inversion. Springer-Verlag, Berlin, Heidelberg, Germany, p. 396.

Kautzleben, H., 1965. Kugelfunktionen. Teubner, Leipzig, Germany, 121 pp. (in German).

Koop, R., 1993. Global Gravity Field Modelling Using Satellite Gravity Gradiometry. Publications on Geodesy, New Series Netherlands Geodetic Commission, No. 38, Delft, The Netherlands, 231 pp.

Metris, G., Xu, J., Wytrzyszczak, 1999. Derivatives of the gravity potential with respect to rectangular coordinates. Celest. Mech. Dyn. Astron. 71, 137–151.

Moritz, H., 1967. Kinematical Geodesy. Report No. 92, Department of Geodetic Science, Ohio State University, Columbus, Ohio, USA.

Moritz, H., 2000. Geodetic reference system 1980. J. Geodesy 74, 128–133.

Pajot, G., de Viron, O., Diament, M., Lequentrec-Lalancette, M.-F., Mikhailov, V., 2008. Noise reduction through joint processing of gravity and gravity gradient data. Geophysics 73, 123–134.

Pavlis, N.K., Holmes, S.A., Kenyon, S.C., Factor, J.K., 2012. The development and evaluation of the Earth Gravitational Model 2008 (EGM2008). J. Geophys. Res. (Solid Earth) 117, B04406 38, pp.

Petrovskaya, M.S., Vershkov, A.N., 2006. Non-singular expressions for the gravity gradients in the local north-oriented and orbital reference frames. J. Geodesy 80, 117–127.

Petrovskaya, M.S., Vershkov, A.N., 2010. Construction of spherical harmonic series for the potential derivatives of arbitrary orders in the geocentric Earth-fixed reference frame. J. Geodesy 84, 165–178.

Rosi, G., Cacciapuoti, L., Sorrentino, F., Menchetti, M., Prevedelli, M., Tino, G.M., 2015. Measurement of the gravity-field curvature by atom interferometry. Phys. Rev. Lett. 114, 013001.

Rummel, R., 1986. Satellite gradiometry. In: Sünkel, H. (Ed.), Mathematical and Numerical Techniques in Physical Geodesy, Lecture Notes in Earth Sciences vol. 7. Springer-Verlag, Berlin, Germany, pp. 317–363.

Rummel, R., 2010. GOCE: Gravitational gradiometry in a satellite. In: Freeden, W., Nashed, M.Z., Sonar, T. (Eds.), Hanbook of Geomathematics. Springer-Verlag, Berlin, Germany, pp. 93–103.

Rummel, R., van Gelderen, M., Koop, R., Schrama, E., Sansò, F., Brovelli, M., Miggliaccio, F., Sacerdote, F., 1993. Spherical Harmonic Analysis of Satellite Gradiometry. Publications on Geodesy, New Series, No. 39, Netherlands Geodetic Commission, Delft, The Netherlands, 124 pp.

Smith, R.S., Thurston, J.B., Dai, T.F., MacLeod, I.N., 1998. *iSPI*™—the improved source parameter imaging method. Geophys. Prospect. 46, 141–151.

Šprlák, M., Novák, P., 2015. Integral formulas for computing a third-order gravitational tensor from volumetric mass density, disturbing gravitational potential, gravity anomaly and gravity disturbance. J. Geodesy 89, 141–157.

Thurston, J.B., Smith, R.S., Guillon, J.C., 2002. A multimodel method for depth estimation from magnetic data. Geophysics 67, 555–561.

Troshkov, G.A., Shalaev, S.V., 1968. Application of the Fourier transform to the solution of the reverse problem of gravity and magnetic surveys. Can. J. Explor. Geophys. 4, 46–62.

Tóth, G., 2005. The gradiometric-geodynamic boundary value problem. In: Jekeli, C., Bastos, L., Fernandes, L. (Eds.), Gravity, Geoid and Space Missions, IAG Symposia Series vol. 129. Springer-Verlag, Berlin, Germany, pp. 352–357.

Veryaskin, A., McRae, W., 2008. On the combined gravity gradient components modeling for applied geophysics. J. Geophys. Eng. 5, 348–356.